

THE $\bar{\partial}$ -EQUATION ON VARIABLE STRICTLY PSEUDOCONVEX DOMAINS

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ABSTRACT. We investigate regularity properties of the $\bar{\partial}$ -equation on domains in a complex euclidean space that depend on a parameter. Both the interior regularity and the regularity in the parameter are obtained for a continuous family of pseudoconvex domains. The boundary regularity and the regularity in the parameter are also obtained for smoothly bounded strongly pseudoconvex domains.

1. INTRODUCTION

We are concerned with regularity properties of the solutions of the $\bar{\partial}$ -equation on the domains D^t that depend on a parameter. We assume that the *total space* $\mathcal{D} := \cup_{t \in [0,1]} D^t \times \{t\}$ is an open subset of $\mathbf{C}^n \times [0, 1]$. Such a family $\{D^t: t \in [0, 1]\}$ is called a *continuous family* of domains D^t in \mathbf{C}^n , or *variable domains* for brevity. Throughout the paper, the parameter t has the range $[0, 1]$ unless specified otherwise.

Let us first introduce Hölder spaces for variable domains. Let $0 \leq \alpha < 1$. Let $\mathbf{N} = \{0, 1, \dots\}$ and $\bar{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$. A family $\{f^t\}$ of functions f^t on D^t is said to be in $C^{\alpha,0}(\mathcal{D})$, if the function $(x, t) \rightarrow f^t(x)$ is continuous on \mathcal{D} and it has finite α -Hölder norms in x variables on compact subsets of \mathcal{D} (see Definition 2.5). By $\{f^t\} \in C_*^{\ell+\alpha,j}(\mathcal{D})$ for $\ell, j \in \bar{\mathbf{N}}$, we mean that all partial derivatives $\partial_x^L \partial_t^i f^t(x)$ are in $C^{\alpha,0}(\mathcal{D})$ for $|L| \leq \ell$ and $i \leq j$. For $k, j \in \bar{\mathbf{N}}$ with $k \geq j$, let $C_*^{k+\alpha,j}(\mathcal{D})$ denote the intersection of all $C_*^{\ell+\alpha,i}(\mathcal{D})$ with $i \leq j, \ell + i \leq k$. Analogously, a family $\{f^t\}$ of $(0, q)$ -forms f^t on D^t is in $C_*^{k+\alpha,j}(\mathcal{D})$ if $\{f_I^t: t \in [0, 1]\}$ are in the space, where coefficients f_I^t are defined in $f^t = \sum f_{i_1 \dots i_q}^t(z) d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{i_q}$ with $i_1 < i_2 < \dots < i_q$.

We say that a *smooth* family $\{D^t\}$ of bounded domains D^t has $C^{k+\alpha,j}$ *boundary*, if D^t admit defining functions r^t on U^t (with $\nabla_x r^t \neq 0$ at $x \in \partial D^t$) such that $\{r^t\} \in C^{k+\alpha,j}(\mathcal{U})$, where $\{U^t\}$ is a continuous family of domains of which the total space \mathcal{U} contains $\bar{\mathcal{D}}$. Finally, a family $\{f^t\}$ of $(0, q)$ -forms f^t with coefficients f_I defined on $\bar{\mathcal{D}}^t$ is of class $C^{k+\alpha,j}(\bar{\mathcal{D}})$, if all partial derivatives $\partial_x^L \partial_t^i f_I^t(x)$, defined on \mathcal{D} , extend continuously to $\bar{\mathcal{D}}$ and their α -Hölder norms on $\bar{\mathcal{D}}$ in x variables are finite.

We will prove the following boundary and interior regularity results.

Theorem 1.1. *Let $1 \leq q \leq n$. Let $k, \ell, j \in \bar{\mathbf{N}}$ satisfy $k \geq j$ and let $0 < \alpha < 1$. Let $\{D^t\}$ be a continuous family of domains in \mathbf{C}^n with an open total space \mathcal{D} in $\mathbf{C}^n \times [0, 1]$. Let $\{f^t\}$ be a family of $\bar{\partial}$ -closed $(0, q)$ -forms f^t on D^t .*

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- (i) Assume that $\{D^t\}$ is a family of bounded domains of $C^{k+2,j}$ boundary, $\{f^t\} \in C^{k+1,j}(\overline{\mathcal{D}})$, and each D^t is strongly pseudoconvex. There exist solutions u^t to $\bar{\partial}u^t = f^t$ on D^t satisfying $\{u^t\} \in C^{k+3/2,j}(\overline{\mathcal{D}})$.
- (ii) Assume that domains D^t respectively admit plurisubharmonic exhaustion functions φ^t with $\{\varphi^t\} \in C^{0,0}(\mathcal{D})$ such that $\{(z, t) \in \mathcal{D} : \varphi^t(z) \leq c\}$ is compact for each $c \in \mathbf{R}$. Assume that $\{f^t\} \in C_*^{\ell+\alpha,j}(\mathcal{D})$ (resp. $C^{k+\alpha,j}(\mathcal{D})$). There exist solutions u^t to $\bar{\partial}u^t = f^t$ on D^t so that $\{u^t\} \in C_*^{\ell+1+\alpha,j}(\mathcal{D})$ (resp. $C^{k+1+\alpha,j}(\mathcal{D})$).

We will call the above $\{\varphi^t\}$ a family of plurisubharmonic *uniform* exhaustion functions of $\{D^t\}$. When $n = 1$, a precise boundary regularity is given by Theorem 4.5. Note that $\{D^t\}$ in (i) satisfies the conditions in (ii). Another example for (ii) is the following \mathcal{D} with rough boundary.

Example 1.2. Let D^t be the ball in \mathbf{C}^n with radius t^{-1} and center $\mathbf{c}(t)$. Let $D^0 = \mathbf{C}^n$. When \mathbf{c} is continuous in $t \in (0, 1]$ and $t^{-1} - |\mathbf{c}(t)| \rightarrow +\infty$ as $t \rightarrow 0^+$, the total space of $\{D^t\}$ is open in $\mathbf{C}^n \times [0, 1]$, while

$$\varphi^t(z) = |z|^2 + \frac{t^2}{1 - t^2|z - \mathbf{c}(t)|^2}$$

are plurisubharmonic uniform exhaustion functions on D^t satisfying $\{\varphi^t\} \in C^{0,0}(\mathcal{D})$.

The study of regularity of solutions of the $\bar{\partial}$ -equation for a fixed domain has a long history. Let us recall some related results. The existence of the smooth solutions on a Stein manifold follows from Dolbeault's theorem and Cartan's Theorem B, as observed by Dolbeault [4]. It is also a main result of the L^2 -theory (see Hörmander [13], [14]). The existence and C^∞ boundary regularity of the canonical solutions for a strongly pseudoconvex compact manifold with smooth boundary were established by Kohn [17] via investigating the $\bar{\partial}$ -Neumann problem; Kohn [18] also obtained C^∞ boundary regularity of possibly non-canonical solutions for a smoothly bounded pseudoconvex domain in \mathbf{C}^n (for instance, see Chen-Shaw [2, p. 122]). The exact regularity in Hölder spaces of $\bar{\partial}$ solutions for $(0, q)$ -forms was obtained by Siu [26] for $q = 1$ and by Lieb-Range [20] for $q \geq 1$.

The domain dependence of the $\bar{\partial}$ -equation has however attracted less attention. The C^∞ regularity of solutions for elliptic partial differential equations on a family of compact complex manifolds (without boundary) was obtained by Kodaira and Spencer [16], which plays an important role in their deformation theory. For planar domains depending on a parameter, the exact regularities of Dirichlet and Neumann problems were obtained recently by Bertrand and Gong [1]. Notice that solving the $\bar{\partial}$ -equation that depends on a parameter has played an important role in the construction of the Henkin-Ramírez functions. However, the domain in this situation is fixed, while multi-parameters enter into the non-homogenous $\bar{\partial}$ -equation. Such parameter dependence is easy to understand once a linear $\bar{\partial}$ -solution operator is constructed. Of course, the construction of such a linear solution operator is included in the classical homotopy formulae; see [23], [7], [11], [19], and [24]. An interesting case is the stability of the $\bar{\partial}$ -equation in terms of a family of strongly pseudoconvex domains; see Greene-Krantz [8]. Their stability results for the $\bar{\partial}$ -solutions can be characterized as the continuous dependence in parameter as defined in section 2. In [3], Diederich-Ohsawa

obtained C^∞ regularity of canonical solutions of the $\bar{\partial}$ -equation for certain smooth $(n, 1)$ -forms. They proved the results via Hörmander's L^2 technique for a family of domains in a Kähler manifold.

Our approach relies essentially on solution operators of the $\bar{\partial}$ -equation that are represented by integral formulae for a smoothly bounded strictly pseudoconvex domain. To deal with variable domains, we will use the Grauert bumps to extend $\bar{\partial}$ -closed forms to a continuous family of larger domains, keeping the forms $\bar{\partial}$ -closed. For a fixed domain, the extension technique is well known through the works of Kerzman [15] and others. To apply the extension for a continuous family of strongly pseudoconvex domains, we will obtain precise regularity results first for a smooth family of strictly convex domains by using the Lieb-Range solution operator [20]. The extension allows us to freeze the domains to apply the classical integral $\bar{\partial}$ -solution operators ([7], [11], [22]). By using a partition of unity in parameter t , we will solve the $\bar{\partial}$ -equation for variable domains with the desired regularity. However, in order to freeze the domains we must restrict them in \mathbf{C}^n . Therefore, there are several questions arising from our approach. For instance, it would be interesting to know if a more general result can be established for the $\bar{\partial}$ -equation on a family of Stein manifolds. We notice a remarkable construction of an integral $\bar{\partial}$ -solution operator by Michel [21] for a smoothly bounded weakly pseudoconvex domain in \mathbf{C}^n . It would be interesting to know if the assertion in Theorem 1.1 (ii) remains true when the domains given are only weakly pseudoconvex.

The paper is organized as follows.

In section 2, we define Hölder spaces for functions on variable domains. In section 3, we adapt Narasimhan lemma and Grauert bumps for variable domains. In section 4, we study the regularity of $\bar{\partial}$ -solutions on variable domains first for strictly convex case and then for strictly pseudoconvex case. The Lieb-Range solution operator and Kerzman's extension method [15] for $\bar{\partial}$ -closed forms are used in section 4 where Theorem 1.1 (i) is proved in Theorem 4.10.

In section 5, we obtain Henkin-Ramírez functions for strictly pseudoconvex open sets that depend on a parameter, which in turn gives us a homotopy formula for variable strictly pseudoconvex domains. The Henkin-Ramírez functions are used in section 6 to obtain a parametrized version of the Oka-Weil approximation. Theorem 1.1 (ii) is proved in Theorem 6.7. As an application, we solve a parametrized version of the Levi problem for variable domains in \mathbf{C}^n . Finally, we use the $\bar{\partial}$ -solutions with parameter to solve a parametrized version of Cousin problems.

2. HÖLDER SPACES FOR FUNCTIONS ON VARIABLE DOMAINS

We first describe some notation used for the rest of the exposition. We use real variables $x = (x_1, \dots, x_d)$ for \mathbf{R}^d . In our application, $d = 2n$. Let ∂_x^k denote the set of partial derivatives in x of order k . Let $\hat{\partial}_x^k$ denote the set of partial derivatives in x of order $\leq k$. Recall that $\mathbf{N} = \{0, 1, 2, \dots\}$, $\bar{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$. Set $\mathbf{R}_+ = [0, \infty)$ and $\bar{\mathbf{R}}_+ = \mathbf{R}_+ \cup \{\infty\}$.

The main purpose of this section is to define Hölder spaces for functions on variable domains. When proving boundary regularity of $\bar{\partial}$ -solutions, we need to parameterize the variable domains up to boundary. Therefore, we will define these spaces first via parametrization. We will then define the spaces without using parametrization. We will discuss the relation between two definitions.

When a is a real number, we denote by $\lfloor a \rfloor$ the largest integer that does not exceed a . Let $D \subset \mathbf{R}^d$ be a bounded domain with C^1 boundary. Let $C^a(\bar{D})$ be the standard Hölder space with norm $|\cdot|_{D;a}$ on \bar{D} . Let $j \geq 0$ be an integer. Let $\{u^t: t_0 \leq t \leq t_1\}$, with t_0, t_1 finite, be a family of functions u^t on \bar{D} . We say that it belongs to $C_*^{a,j}(\bar{D} \times [t_0, t_1])$, abbreviated by $\{u^t\} \in C_*^{a,j}(\bar{D} \times [t_0, t_1])$, if $t \mapsto \partial_t^i u^t$, with $i \leq j$, are continuous maps from $[t_0, t_1]$ into $C^{\lfloor a \rfloor}(\bar{D})$, and if they are bounded maps sending $[t_0, t_1]$ into $C^{a-\lfloor a \rfloor}(\bar{D})$. For a real number a and an integer j with $a \geq j \geq 0$, define

$$C^{a,j}(\bar{D} \times [t_0, t_1]) = \bigcap_{i \leq j} C_*^{a-i,i}(\bar{D} \times [t_0, t_1]).$$

For brevity, we write $C_*^{b,j}(\bar{D}) = C_*^{b,j}(\bar{D} \times [0, 1])$ and $C^{a,j}(\bar{D}) = C^{a,j}(\bar{D} \times [0, 1])$.

We now define Hölder spaces on variable domains given by a parametrization. Let D^t be domains in \mathbf{R}^d and let Γ^t be C^1 embeddings from \bar{D} onto \bar{D}^t . Let $\{u^t\}$ be a family of functions u^t , which are respectively defined on D^t . Denote by $\{u^t\} \in C_*^{a,j}(\bar{D}_\Gamma)$ when $\{v^t \circ \Gamma^t\}$ is in $C_*^{a,j}(\bar{D})$. Define

$$C^{a,j}(\bar{D}_\Gamma) = \bigcap_{i=0}^j C_*^{a-i,i}(\bar{D}_\Gamma),$$

for an integer j in $[0, a]$. We define $C_*^{\infty,j}(\bar{D}_\Gamma) = \bigcap_{k=1}^{\infty} C_*^{k,j}(\bar{D}_\Gamma)$. Similarly, define $C^{\infty,j}(\bar{D}_\Gamma)$, $C_*^{\infty,\infty}(\bar{D}_\Gamma)$, and $C^{\infty,\infty}(\bar{D}_\Gamma)$.

While writing $\{u^t\}$ as u and $\{v^t \circ \Gamma^t\}$ as $v \circ \Gamma$, we define the norms:

$$\begin{aligned} |u|_{D;a,i} &= \sup_{0 \leq \ell \leq i, t \in [0,1]} \{|\partial_t^\ell u^t|_{D;a}\} \quad \text{if } u \in C_*^{a,i}(\bar{D}); \\ \|u\|_{D;a,j} &= \max_{0 \leq i \leq j} \{|u|_{D;a-i,i}\}, \quad \text{if } u \in C^{a,i}(\bar{D}), \\ |v|_{D_\Gamma;a,j} &= |v \circ \Gamma|_{D;a,j}, \quad \text{if } v \in C_*^{a,j}(\bar{D}_\Gamma); \\ \|v\|_{D_\Gamma;a,j} &= \|v \circ \Gamma\|_{D;a,j}, \quad \text{if } v \in C^{a,j}(\bar{D}_\Gamma). \end{aligned}$$

Note that when defining $C^{a,j}(\bar{D}_\Gamma)$, we assume that D is bounded with C^1 boundary and $a \geq j$. Let us first ensure the independence of the Hölder spaces on the parametrization under mild conditions.

Lemma 2.1. *Let D, D^t be bounded domains in \mathbf{R}^d with C^1 boundary, and let $\{\Gamma^t\} \in C^{a,j}(\bar{D}) \cap C^{1,0}(\bar{D})$ be a family of embeddings Γ^t from \bar{D} onto \bar{D}^t . Then we have the following.*

- (i) $\mathcal{D} = \cup_t D^t \times \{t\}$ is open in $\mathbf{R}^d \times [0, 1]$, $\bar{\mathcal{D}}$ is compact, and $\bar{\mathcal{D}} = \cup_t \bar{D}^t \times \{t\}$.
- (ii) A family $u = \{u^t\}$ of functions u^t on \bar{D}^t is in $C^{a,j}(\bar{D}_\Gamma)$ if and only if

$$\begin{aligned} \partial_t^\ell \partial_x^K u^t(x) &\in C^0(\bar{\mathcal{D}}), \quad \forall \ell \leq j, |K| \leq a - \ell; \\ \|u\|_{\mathcal{D};a,j} &:= \max_{0 \leq i \leq j} \{|u|_{\mathcal{D};a-i,i}\} < \infty \end{aligned}$$

with $|u|_{\mathcal{D};b,i} := \sup_{0 \leq \ell \leq i, t \in [0,1]} \{|\partial_t^\ell u^t|_{D^t;b}\}$. For some constants C_1, C_2 depending on $\|\{\Gamma^t\}\|_{D;a,j}$ and $\inf_{\mathcal{D}} |\partial_x \Gamma^t|$

$$C_1^{-1} \|u\|_{D_\Gamma;a,j} \leq \|u\|_{\mathcal{D};a,j} \leq C_2 \|u\|_{D_\Gamma;a,j}.$$

Proof. We get (i) easily, since $(x, t) \mapsto (\Gamma^t(x), t)$ defines a homeomorphism Γ sending $\overline{D} \times [0, 1]$ onto $\cup_t \overline{D^t} \times \{t\}$. Thus $\mathcal{D} = \Gamma(D \times [0, 1])$ is open in $\mathbf{R}^d \times [0, 1]$ and $\partial \mathcal{D} = \Gamma(\partial D \times [0, 1])$ is compact.

Throughout the paper, the boundary value of a partial derivative $\partial_t^\ell \partial_x^K u^t$ at a point in $\partial \mathcal{D}$ is regarded as an extension in the pointwise limit, if it exists, of the derivatives in the open set \mathcal{D} .

We now verify (ii). Since Γ^t are embeddings and $\{\Gamma^t\} \in C^{1,0}(\overline{D})$, the Jacobian matrix $\partial_x \Gamma^t$ is non-singular and continuous on $\overline{D} \times [0, 1]$. Since $\partial D \in C^1$, the fundamental theorem of calculus yields

$$|(\partial_x \Gamma^t)(x' - x)|/2 \leq |\Gamma^t(x') - \Gamma^t(x)| \leq 2|(\partial_x \Gamma^t)(x' - x)|,$$

when x' is sufficiently close to x . This shows that

$$|x' - x|/C \leq |\Gamma^t(x') - \Gamma^t(x)| \leq C|x' - x|.$$

Thus, we obtain the lemma for $a < 1$. Let $y = \Gamma^t(x)$. By the chain rule, we have

$$\partial_y((\Gamma^t)^{-1}) = (\partial_x \Gamma^t)^{-1} \circ (\Gamma^t)^{-1}(y), \quad \partial_t(\Gamma^t)^{-1}(y) = -((\partial_x \Gamma^t)^{-1} \partial_t \Gamma^t) \circ (\Gamma^t)^{-1}(y).$$

Assume that $\{u^t\} \in C^{1,0}$. Since $\{\Gamma^t\} \in C^{1,0}$, the chain rule says that

$$(\partial_y u^t) \circ \Gamma^t \partial_x \Gamma_1^t = \partial_x(u^t \circ \Gamma^t(x)).$$

This shows that $\{u^t\}, \{\partial_y u^t\}$ are in $C^{0,0}$ if and only if $\{u^t\} \in C^{1,0}$. Assume that $\{u^t\} \in C^{1,1}$. Using the existence of partial derivatives $\partial_y u^t$ and computing limits directly, we verify that

$$(\partial_t u^t) \circ \Gamma^t(x) = \partial_t(u^t \circ \Gamma^t(x)) - (\partial_y u^t) \circ \Gamma^t(x) \partial_t \Gamma^t(x)$$

for $x \in D$ and hence for all $x \in \overline{D}$ by continuity. This shows that $\{u^t\}, \{\partial_y u^t\}$, and $\{\partial_t u^t\}$ are in $C^{0,0}$ if and only if $\{u^t\}$ is in $C^{1,1}$. The lemma follows immediately for other values of a, j . \square

We now define $C_*^{a,j}(D_\Gamma)$ for a family of C^1 embeddings Γ^t from an arbitrary open set $D \subset \mathbf{R}^d$ onto $D^t \subset \mathbf{C}^n$. By $\{f^t\} \in C_*^{a,j}(D_\Gamma)$, we mean that f^t are functions on D^t such that $\{f^t\} \in C_*^{a,j}(K_\Gamma)$ for any subset K of D with smooth boundary. Again, when all Γ^t are the identity map, we define $C_*^{a,j}(D)$ to be $C_*^{a,j}(D_\Gamma)$. Define $C^{a,j}(D_\Gamma)$ and $C^{a,j}(D)$ similarly.

We will denote by $\mathcal{U}(E)$ a neighborhood of E when E is a subset of \mathbf{R}^d . For example, we say that D is defined by $r < 0$ if r is a real function on some $\mathcal{U}(\overline{D})$ on which D is defined by $r < 0$.

We will use the following Seeley extension operator [25].

Lemma 2.2. *Let $H = \mathbf{R}^d \times [0, \infty)$. There is a continuous linear extension operator $E: C_0^0(H) \rightarrow C_0^0(\mathbf{R}^{d+1})$ such that $Ef = f$ on H and $E: C_0^a(H) \rightarrow C_0^a(\mathbf{R}^{d+1})$ is continuous for each $a \geq 0$.*

Here C_0^\bullet stands for functions with compact support. Seeley [25] showed that there are numerical sequences $\{a_k\}, \{b_k\}$ such that (i) $b_k < 0$ and $b_k \rightarrow -\infty$, $(-1)^k a_k > 0$, (ii) $\sum |a_k| \cdot |b_k|^n < \infty$ for $n = 0, 1, 2, \dots$, (iii) $\sum a_k (b_k)^n = 1$ for $n = 0, 1, 2, \dots$. Then Seeley defined the extension

$$(Ef)(x, s) = \sum_{k=0}^{\infty} a_k \phi(b_k s) f(x, b_k s).$$

Here ϕ is a C^∞ function satisfying $\phi(s) = 1$ for $s < 1$ and $\phi(s) = 0$ for $s > 2$. For a differential form f , we define Ef by extending the coefficients of f via E .

Let us use the extension operator to discuss the space $C^{a,j}(\overline{D}_\Gamma)$ and a version of extension for variable domains. We will also discuss an approximation. Let $C_0^{a,j}(D_\Gamma)$ denote the set of $\{f^t\} \in C^{a,j}(D_\Gamma)$ such that $(x, t) \rightarrow f^t(x)$ has compact support in the total space of $\{\Gamma^t(D)\}$.

Lemma 2.3. *Let D be a bounded domain in \mathbf{R}^d with C^a boundary and $a \geq 1$. Let $\{\Gamma^t\} \in C^{a,j}(\overline{D})$ be a family of embeddings Γ^t from \overline{D} onto \overline{D}^t .*

(i) *There exists an open neighborhood U of \overline{D} with $\partial U \in C^a$ and a family of embeddings $\hat{\Gamma}^s$ from \overline{U} onto \overline{U}^s for $s_0 \leq s \leq s_1$ with $s_0 < 0$, $1 < s_1$ such that $\{\hat{\Gamma}^s: s_0 \leq s \leq s_1\}$ is of class $C^{a,j}(\overline{U})$. Moreover, $\hat{\Gamma}^t = \Gamma^t$ on \overline{D} for $t \in [0, 1]$. There exists an \mathbf{R} -linear extension operator*

$$E: C^{a,j}(\overline{D}_\Gamma) \rightarrow C_0^{a,j}(U_{\hat{\Gamma}})$$

satisfying $\|Ef\|_{U_{\hat{\Gamma}}; a', j'} \leq C_a \|f\|_{D_\Gamma; a', j'}$ for all finite $a' \leq a$ and finite $j' \leq j$. Here

$$(Ef)^t|_{\overline{D}^t} = f^t, \quad f = \{f^t\}, \quad \forall t \in [0, 1].$$

(ii) *Let $f \in C^{b,k}(\overline{D}_\Gamma)$. There exists a sequence in $C^{a,j}(\overline{D}_\Gamma)$ that is bounded in $C^{b,k}(\overline{D}_\Gamma)$ and converges to f in $C^{[b],k}(\overline{D}_\Gamma)$.*

Remark 2.4. In (i) of the lemma, the $C^{a,j}(\overline{U}_{\hat{\Gamma}})$ is defined as $C^{a,j}(\overline{U}_\Gamma)$, where $\{\Gamma^t: 0 \leq t \leq 1\}$ is replaced by $\{\hat{\Gamma}^s: s_0 \leq s \leq s_1\}$.

Proof. (i) Since ∂D is in C^a with $a \geq 1$, we can locally use a C^a coordinate change φ to transform ∂D into the boundary of the half-space $x_d \geq 0$. We then extend the mapping $f^t := \Gamma^t \circ \varphi^{-1}$ to g^t by

$$(2.1) \quad g^t(x) = \sum_{k=0}^{\infty} a_k \phi(b_k x_d) f^t(x', b_k x_d), \quad \forall x_d < 0.$$

Set $\Gamma_*^t = g^t \circ \varphi$. Thus, using a partition of unity and local changes of coordinates, we can extend $\Gamma^t: \overline{D} \rightarrow \overline{D}^t$ to embeddings Γ_*^t from \overline{U} onto \overline{U}^t for a smoothly bounded domain U containing \overline{D} , while $\{\Gamma_*^t\} \in C^{a,j}(\overline{U})$. Next, we extend $\{\Gamma_*^t: 0 \leq t \leq 1\}$ to a family $\{\hat{\Gamma}^s: s_0 \leq s \leq s_1\}$. Let us use Seeley's extension for the half-space $s \geq 0$. By a partition of unity in the t variable, we may assume that $\Gamma_*^t = 0$ for $1/2 < t < \infty$. Define

$$(2.2) \quad \hat{\Gamma}^s(x) = \sum_{\ell=0}^{\infty} a_\ell \phi(b_\ell s) \Gamma_*^{b_\ell s}(x), \quad \forall s < 0.$$

Applying Seeley's extension to the half-space $s \leq 1$, we can extend $\{\Gamma^t\}$ to a family of embeddings $\hat{\Gamma}^s$ from \overline{U} onto $\overline{U^s}$ for $s \in [s_0, s_1]$, when $-s_0, s_1 - 1$ are positive and sufficiently small. We leave it to the reader to check that $\{\hat{\Gamma}^s\}$ is in $C^{a,j}(\overline{U})$. The extension $E: C^{a,j}(\overline{D}_\Gamma) \rightarrow C^{a,j}(\overline{U}_{\hat{\Gamma}})$ can be defined by formulas similar to (2.1)-(2.2) and by shrinking \overline{U} , $[s_0, s_1]$ slightly.

(ii) As above, a family $\{f^t\} \in C^{b,k}(\overline{D})$ of functions f^t on $\overline{D^t}$ extends to a family $\{\tilde{f}^t\} \in C^{b,k}(\overline{U}_{\hat{\Gamma}})$. Let $\{f^t\} \in C^{b,k}(\overline{D})$. Applying the standard smoothing operator on $U \times (s_0, s_1)$ to $\tilde{f}^s(\hat{\Gamma}^s(x))$, we can verify the approximation on the compact subset $\overline{D} \times [0, 1]$ of $U \times (s_0, s_1)$. \square

Lemma 2.1 says that with $a \geq 1$, space $C^{a,j}(\overline{D}_\Gamma)$ does not depend on the parameterizations $\Gamma^t: \overline{D} \rightarrow \overline{D^t}$, provided they exist. Next, we study the existence of parameterizations Γ^t . To this end, we first introduce function spaces without using parametrization. Recall from section 1 that $\{D^t: t \in [0, 1]\}$ a continuous family of domains D^t in \mathbf{R}^d , if $\mathcal{D} = \cup_t D^t \times \{t\}$ is open in $\mathbf{R}^d \times [0, 1]$. The \mathcal{D} is the total space of the family $\{D^t\}$.

Definition 2.5. (i) Let $\{D^t: t_0 \leq t \leq t_1\}$ be a continuous family of domains in \mathbf{R}^d with total space \mathcal{D} . We say that a family $\{u^t\}$ of functions u^t on $\overline{D^t}$ is in $C_*^{b,j}(\overline{\mathcal{D}})$ for finite b and j , if

$$\begin{aligned} \partial_t^\ell \partial_x^L u^t(x) &\in C^0(\overline{\mathcal{D}}), \quad \forall \ell \leq j, |L| \leq b; \\ |u|_{\mathcal{D};b,i} &:= \sup_{0 \leq \ell \leq i, t \in [t_0, t_1]} \{|\partial_t^\ell u^t|_{D^t;b}\} < \infty. \end{aligned}$$

Define $C^{a,j}(\overline{\mathcal{D}}) = \bigcap_{i \leq j} C_*^{a-i,i}(\overline{\mathcal{D}})$ and the norm

$$\|u\|_{\mathcal{D};a,j} := \max_{0 \leq i \leq j} \{ |u|_{\mathcal{D};a-i,i} \}.$$

Define $C_*^{b,j}(\overline{\mathcal{D}})$, $C^{a,j}(\overline{\mathcal{D}})$ similarly if one of a, b, j is infinity. Let $C_*^{b,j}(\overline{\mathcal{D}})$, $C^{a,j}(\overline{\mathcal{D}})$ be the sets of functions v on $\overline{\mathcal{D}}$ with $\{v(\cdot, t)\}$ in $C_*^{b,j}(\overline{\mathcal{D}})$, $C^{a,j}(\overline{\mathcal{D}})$, respectively.

(ii) By $\{u^t\} \in C^{a,j}(\mathcal{D})$ we mean that $\{u^t\} \in C^{a,j}(\overline{\omega})$ for any relatively compact open subset ω of \mathcal{D} and for $\omega^t = \{x: (x, t) \in \omega\}$. Define $C_*^{b,j}(\mathcal{D})$, $C^{a,j}(\mathcal{D})$, $C_*^{b,j}(\mathcal{D})$, $C^{a,j}(\mathcal{D})$ analogously. The topology on $C^{a,j}(\mathcal{D})$ is defined by semi-norms $\|\cdot\|_{\omega;a',j'}$, where $a' \leq a, j' \leq j$, a', j' are finite the sets ω are relatively compact in \mathcal{D} . Define the topologies of other spaces analogously.

For clarity, sometimes we denote $C^{a,j}(\mathcal{D})$, $C^{a,j}(\overline{\mathcal{D}})$ by $C^{a,j}(\{D^t\})$, $C^{a,j}(\{\overline{D^t}\})$, respectively.

Proposition 2.6. Let $a \in \overline{\mathbf{R}}_+, j \in \overline{\mathbf{N}}$ and $a \geq j$. Let $\{D^t\}$ be a continuous family of domains in \mathbf{R}^d with a bounded total space \mathcal{D} and let $\partial\mathcal{D}$ be the boundary of \mathcal{D} in $\mathbf{R}^d \times [0, 1]$. Then the following are equivalent:

- (i) For each $t_0 \in [0, 1]$, there exist a neighborhood I of t_0 in $[0, 1]$ and a family $\{\Gamma^t\} \in C^{a,j}(\overline{D} \times \overline{I}) \cap C^{1,0}(\overline{D} \times \overline{I})$ of embeddings Γ^t from \overline{D} onto $\overline{D^t}$ with $\partial D \in C^a \cap C^1$.
- (ii) $\overline{\mathcal{D}}$ is compact. For every $(x_0, t_0) \in \partial\mathcal{D}$ there exist an open neighborhood ω of (x_0, t_0) in $\mathbf{R}^d \times [0, 1]$ and a real function $r \in C^{a,j}(\overline{\omega}) \cap C^{1,0}(\overline{\omega})$ such that

$$(2.3) \quad \mathcal{D} \cap \omega = \{(x, t) \in \omega: r(x, t) < 0\}; \quad \nabla_x r(x, t) \neq 0, \quad \forall (x, t) \in \partial\mathcal{D} \cap \omega.$$

(iii) There is a bounded domain ω in $\mathbf{R}^d \times [0, 1]$ with $\overline{\mathcal{D}} \subset \omega$ and a real function r on ω of class $C^{a,j} \cap C^{1,0}$ such that (2.3) holds.

We say that $\{D^t\}$ is a smooth family of bounded domains of $C^{a,j} \cap C^{1,0}$ boundary if it satisfies one of the above equivalent conditions.

Proof. We first show that (i) implies (iii). Assume that such a family $\{\Gamma_\alpha^t\}$ exists for $t \in \overline{I_\alpha}$, where I_α is a connected open set I_α in $[0, 1]$ and $\{I_\alpha\}$ is a finite open covering of $[0, 1]$. Since $[0, 1]$ is connected, we may assume that D is independent of α . By Lemma 2.3 (i), we may assume that Γ_α^t extend to embeddings Γ_α^t from \overline{U} onto $\overline{U^t}$ with $\overline{D} \subset U$ and $\{\Gamma_\alpha^t\} \in C^{a,j}(\overline{U} \times \overline{I_\alpha})$. Let r_0 be a real function of class C^a on U such that D is defined by $r_0 < 0$ and $\nabla r_0 \neq 0$ on ∂D . Then $r(x, t) = \sum \chi_\alpha(t) r_0 \circ (\Gamma_\alpha^t)^{-1}(x)$ has the desired property, while ω is an open subset of $\bigcup_\alpha \{(\Gamma_\alpha^t(x), t) : x \in U\}$.

Clearly, (iii) implies (ii). Let us now show that (ii) implies (iii). To this end, let us show that there are a neighborhood ω of $\overline{\mathcal{D}}$ and a real function $r \in C^{a,j}(\overline{\omega})$ such that (2.3) holds. In other words, r is a global definition function of \mathcal{D} . By (2.3), it is clear that \mathcal{D} does not contain boundary points of \mathcal{D} . So \mathcal{D} is open. Since $\partial \mathcal{D}$ is compact, we cover it by finitely many open sets ω_α such that on ω_α there are real functions $r_\alpha \in C^{a,j}(\overline{\omega_\alpha})$ such that

$$\mathcal{D} \cap \omega_\alpha : r_\alpha < 0; \quad \nabla_x r_\alpha(x, t) \neq 0, \quad \forall x \in \partial D^t.$$

Let $r_0 = -1$ on $\omega_0 = \mathcal{D}$. Choose a partition $\{\chi_0, \chi_\alpha\}$ of unity by non-negative functions subordinate to the open covering $\{\omega_0, \omega_\alpha\}$. Note that $\nabla_x r_\alpha(x, t)$ is a non-zero outward normal vector at $x \in \partial D^t$. Shrinking $\omega = \omega_0 \cup (\bigcup_\alpha \omega_\alpha)$ slightly, we can verify that $r = \sum \chi_\alpha r_\alpha$ has the required property.

Next, let us show that (iii) implies (i). Assume that the function r is given as in (2.3) with ω being an open neighborhood of $\overline{\mathcal{D}}$. Define $\omega^t = \{x : (x, t) \in \omega\}$. Fix t_0 . We need to find a family of embeddings Γ^t from \overline{D} onto $\overline{D^t}$ for t near t_0 . Let $n(x)$ be the gradient vector field of $r(x, t_0)$. We approximate $n(x)$ by a vector field $v(x)$ such that v is of class C^a on ω^{t_0} and such that for x in a small neighborhood V of ∂D^{t_0} , the line segment $\{x + sv(x) : -\epsilon \leq s \leq \epsilon\}$ intersects ∂D^t transversally at a unique point with $s = S(x, t)$ for $|t - t_0| < \delta$. Note that s is the unique solution to

$$r(x + sv(x), t) = 0, \quad |s| < \epsilon.$$

Let $D_c^t \subset \omega^t$ be defined by $r(\cdot, t) < c$. Fix a small positive constant $-c_1$. For x near $\partial D_{c_1}^{t_0}$, let $b \in [-\epsilon, \epsilon]$ be the unique number such that $x + bv(x)$ is in ∂D^{t_0} . For t close to t_0 , there exists a unique $\tilde{b} \in [-\epsilon, \epsilon]$ such that $x + \tilde{b}v(x)$ is in ∂D^t . Note that b depends on x , while \tilde{b} depends on x, t ; and both are positive. We will find $\nu = \nu(x, t, \lambda)$ that is strictly increasing in λ such that $\nu(x, t, \lambda) \equiv \lambda$ for λ near 0, while at the end point $\lambda = b(x)$ we have

$$\nu(x, t, b(x)) = \tilde{b}(x, t).$$

For the existence of ν and its smoothness, we take a smooth decreasing function χ such that $\chi(\lambda)$ equals 1 near $\lambda = 0$ and 0 near $\chi = b$. Furthermore, $\int_0^b \chi d\lambda < b/2$. Note that the latter is less than \tilde{b} when t is close to t_0 . Define

$$(2.4) \quad \chi_1 = \frac{\tilde{b}(x, t) - \int_0^b \chi d\lambda}{\int_0^b (1 - \chi) d\lambda} (1 - \chi), \quad \nu = \int_0^\lambda (\chi + \chi_1) d\lambda.$$

We then define $\Gamma^t = \text{I}$ on $D_{c_1}^{t_0}$ and

$$\Gamma^t(x + \lambda v(x)) = x + \nu(x, t, \lambda)v(x), \quad \text{for } 0 \leq \lambda \leq b(x), x \in \partial D_{c_1}^{t_0}.$$

Then Γ^t embeds $\overline{D^{t_0}}$ onto $\overline{D^t}$ for t close to t_0 . To verify the smoothness of $\{\Gamma^t\}$, let x, y be in \mathbf{R}^d . We start with equations that determine $b = b(x)$:

$$x + bv(x) = y, \quad r(y, t_0) = 0, \quad r(x, t_0) = c_1.$$

The first two equations determine b via x . Indeed, the Jacobian determinant of $y - x - bv(x), r(y, t_0)$ in y, b equals $-v(x) \cdot \nabla_y r(y, t_0)$. The latter is not zero, since x is close to y , $v(x)$ is close to $\nabla_y r(y, t_0)$, and $\nabla_y r(y, t_0) \neq 0$ near ∂D^{t_0} . This shows that b is a function of class C^a in x near ∂D^{t_0} . The $\tilde{b} = \tilde{b}(y, t)$ is determined by

$$x + \tilde{b}v(x) = y, \quad r(y, t) = 0, \quad r(x, t_0) = c_1.$$

We see that \tilde{b} is of class $C^{a,j}$ in y near ∂D^{t_0} and in t near t_0 . Finally, we consider equations $\Gamma^t(y) = u(y, t)$, which can be written as

$$x + \lambda v(x) = y, \quad r(x, t_0) = c_1, \quad u = x + \nu(x, t, \lambda)v(x).$$

We want to use the first two equations to determine x, λ via y and the last equation to determine $u(y, t)$. Recall that $0 \leq \lambda \leq b(x) \leq \epsilon$ and $\epsilon > 0$ is small. At $\lambda = 0$, the Jacobian determinant of $x + \lambda v(x), r(x, t_0)$ in x, λ is $-|\nabla_x r(x, t_0)|^2$, which is non zero. By the implicit function theorem, we can verify that (x, λ) is of class C^a in y . By the smoothness properties of b, \tilde{b} verified above and by (2.4), we know that $\nu(x, t, \lambda)$ is of class $C^{a,j}$ in y, t . This shows that $\{\Gamma^t\}$ is of class $C^{a,j}$ as t varies near t_0 . \square

Remark 2.7. By Lemma 2.1 and Proposition 2.6, the $C^{a,j}(\overline{D}_\Gamma)$ space is independent of $\{\Gamma^t\} \in C^{a,j}(\overline{D})$ when $a \geq 1$. Furthermore, the parameter t can be in several variables and a parametrization Γ^t can be obtained for t near a given point t_0 .

The smooth approximation for $C^{a,j}(\overline{D})$ is given by Lemma 2.2. We conclude this section with the following approximation result.

Proposition 2.8. *Let $\{D^t\}$ be a continuous family of domains. Then $C^{\infty,\infty}(\mathcal{D})$ is dense in $C^{a,j}(\mathcal{D})$ and in $C_*^{b,j}(\mathcal{D})$.*

Proof. We know that \mathcal{D} is open in $\mathbf{R}^d \times [0, 1]$. If D^0 is non-empty, we extend D^t to a larger family by setting $D^t = D^{-t}$ for $-1 < t < 0$; if D^1 is non-empty, we set $D^{2-t} = D^t$ for $0 < t < 1$. Then the total space of the extended family is open in $\mathbf{R}^d \times (-1, 2)$. Using partition of unity and Seeley's extension, we can extend each $\{f^t\} \in C_*^{b,j}(\mathcal{D})$ to a family $\{\tilde{f}^t\}$ in $C_*^{b,j}(\{D^t: -1 < t < 2\})$. By the standard smoothing, we can get the approximation. \square

We have provided necessary background for Hölder spaces on variable domains. In our applications, boundary regularity for the \overline{D} -solutions will be derived only in $C^\bullet(\overline{D})$ spaces, while the proof of interior regularity is more flexible and it will be carried out in $C_*^\bullet(\mathcal{D})$ and $C^\bullet(\mathcal{D})$.

3. NARASIMHAN LEMMA AND GRAUERT BUMPS FOR VARIABLE DOMAINS

The main purpose of this section is to recall a construction of Grauert's bumps. We will provide precise smoothness for the bumps with a parameter, which are needed for us to understand the boundary regularity of $\bar{\partial}$ -solutions on variable domains in section 4.

We need some facts about defining functions of a domain. A bounded domain D in \mathbf{C}^n that has C^2 boundary is defined by $r < 0$, where r is a C^2 defining function defined near \bar{D} and $\nabla r \neq 0$ on ∂D . Then the defining function $\tilde{r} = e^{Lr} - 1$ enjoys further properties. Assume that L is sufficiently large. When D is strictly pseudoconvex, the complex Hessian of \tilde{r} is positive definite near ∂D . Note that each connected component of D is strictly convex if and only if the real Hessian Hr is positive definite on the tangent space of ∂D . The latter implies that $H\tilde{r}$ is positive definite at each point of ∂D . Finally, D is strictly convex if and only if D is connected and $H\tilde{r}$ is positive-definite at each point of ∂D , equivalently if

$$\operatorname{Re}\{\tilde{r}_\zeta \cdot (\zeta - z)\} \geq |\zeta - z|^2/C, \quad \forall \zeta \in \partial D, z \in \bar{D},$$

for some positive number C . In our proofs, a domain may not be connected, while a convex domain is always connected.

Lemma 3.1. *Let $j \in \overline{\mathbf{N}}$ and $a \in \overline{\mathbf{R}}_+$ with $j \leq a$. Let $\{D^t\}$ be a smooth family of bounded domains in \mathbf{C}^n with $C^{a+2,j}$ boundary. Assume that D^t are strictly pseudoconvex. For each $t_0 \in [0, 1]$, there are an open neighborhood I of t_0 , a connected open neighborhood U of \bar{D} with $\partial U \in C^{a+2}$, biholomorphic mappings ψ_i from ω_i onto B_{ϵ_0} , and smooth families $\{D_i^t\}, \{N_i^t\}, \{B_i^t\}$ of bounded domains of $C^{a+2,j}$ boundaries satisfying the following:*

(i) *The $\hat{N}_i^t := \psi_i(N_i^t)$ are strictly convex and relatively compact in B_{ϵ_0} , and*

$$(3.1) \quad \begin{aligned} D_{i+1}^t &= D_i^t \cup B_i^t, \quad D_0^t = D^t, \quad \bar{D}^t \subset D_m^t, \quad m < \infty; \\ N_i^t &= D_i^t \cap B_i^t, \quad (\overline{B_i^t \setminus D_i^t}) \cap (\overline{D_i^t \setminus N_i^t}) = \emptyset. \end{aligned}$$

(ii) *There exists a family $\{\Gamma^t\} \in C^{a+2,j}(\bar{U})$ of embeddings Γ^t from $\bar{U} \rightarrow \bar{U}^t$ such that $\overline{D_i}, \overline{B_i}, \overline{N_i}$ are compact in U and*

$$\Gamma^t(\overline{D_i}) = \overline{D_i^t}, \quad \Gamma^t(\overline{B_i}) = \overline{B_i^t}, \quad \Gamma^t(\overline{N_i}) = \overline{N_i^t}.$$

(iii) *Each D_i^t (resp. \hat{N}_i^t) is defined by $r_i^t < 0$ on U (resp. $\hat{r}_i^t < 0$ on B_{ϵ_0}), r_i^t is strictly plurisubharmonic near $\bar{U} \setminus D_i^t$, and \hat{r}_i^t is strictly convex on B_{ϵ_0} . Furthermore, $\{\hat{r}_i^t\} \in C^{a+2,j}(\overline{B_{\epsilon_0}} \times \bar{I})$ and $\{r_i^t\} \in C^{a+2,j}(\bar{U} \times \bar{I})$.*

Proof. We will first construct N and B for a fixed domain D by some local changes of coordinates.

We assume that the defining function r of D is C^{a+2} and strictly plurisubharmonic near ∂D . Fix a point $p \in \partial D$. We want to construct a bump B containing p and a biholomorphic mapping ψ defined on \bar{B} such that $\psi(\bar{N})$ is strictly convex for $N = B \cap D$. Furthermore, $D_1 = D \cup B$, B , and N are strictly pseudoconvex with C^{a+2} boundary. We also require that $(\overline{B \setminus D}) \cap (\overline{D \setminus N}) = \emptyset$.

More precisely, let us choose a unitary matrix S such that the map $\varphi_0: z \mapsto S(z - p)$ sends the inner normal vector of ∂D at p to the y_n -axis. We will apply two more changes of

coordinates that are uniquely determined by Taylor coefficients of $r_1(z) := \frac{1}{2|r_z(p)|} r \circ \varphi_0^{-1}(z)$ at the origin. We then specify B and N .

Assume that φ_0 has been determined. Near the origin, $D' = \varphi_0(D)$ is defined by $r_1 < 0$ with $r_1(z) = -y_n + O(2)$. In the Taylor polynomial, we have

$$r_1 = -y_n + \operatorname{Re} \sum a_{ij} z_i z_j + \sum b_{i\bar{j}} z_i \bar{z}_j + h_1(z)$$

with $h_1(z) = o(2)$. Define a coordinate transformation $\tilde{z} = \varphi_1(z)$ by $\tilde{z}' = z'$ and

$$\tilde{z}_n = z_n - i \sum a_{jk} z_j z_k - i b_{n\bar{n}} z_n z_n - i \sum_{\alpha} b_{\alpha\bar{n}} z_{\alpha} z_n.$$

Then $D'' = \varphi_1(\mathcal{U}(0) \cap D')$ is defined by $r_2 < 0$ for $r_2 := r_1 \circ \varphi_1^{-1}$. We have

$$r_2 = -y_n + \sum_{\alpha, \beta=1}^{n-1} b_{\alpha\bar{\beta}} z_{\alpha} \bar{z}_{\beta} + h_2(z)$$

and $h_2(z) = o(2)$. Define $\tilde{z} = \varphi_2(z)$ by $\tilde{z}_n = z_n - i z_n^2$ and $\tilde{z}' = z'$. Then $D''' = \varphi_2(\mathcal{U}(0) \cap D'')$ is defined by $r^* < 0$ with $r^* = e^{r_2 \circ \varphi_2^{-1}} - 1$ and

$$r^* = -y_n + |z_n|^2 + \sum_{\alpha, \beta=1}^{n-1} b_{\alpha\bar{\beta}} z_{\alpha} \bar{z}_{\beta} + h(z), \quad h(z) = o(2).$$

Obviously, r^* is C^{a+2} and strictly convex on some $\overline{B_{\epsilon_0}}$.

Let χ_0 be a smooth convex function vanishing solely on $(-\infty, 1]$. Let

$$\hat{N} \subset B_{\epsilon_0} : \hat{r}(z) := r^*(z) + C^* \chi_0(\epsilon_1^{-2} |z|^2) < 0.$$

For $C^* > 0$ sufficiently large, \hat{r} is strictly convex on B_{ϵ_0} and \hat{N} is connected and relatively compact in B_{ϵ_0} . We remark that φ_i depend on the first and second-order derivatives of r at p , while ϵ_0 and ϵ_1 depend on the least upper bound of $|\partial_z r(p)|^{-1}$ as well as on the norms of the first and second order derivatives of r . Define $\psi = \varphi_2 \varphi_1 \varphi_0$ and $N = \psi^{-1}(\hat{N})$.

Let χ_1 be a smooth function on \mathbf{R} that is 1 for $|t| < 1$ and 0 for $|t| > 2$. Define

$$\tilde{D} : \tilde{r}(z) := r(z) - \delta \chi_1(\epsilon_2^{-2} |z - p|^2) < 0; \quad B = N \cup (\tilde{D} \setminus D).$$

Here $0 < \epsilon_2 < \epsilon_1 / C_*$ for some C_* that depends only on the least upper bound of $|\partial_z r(p)|^{-1}$ and the norms of the first two derivatives of r ; and $\psi(B_{\epsilon_2}(p))$ is contained in $B_{\epsilon_1}(0)$. When $\delta > 0$ is sufficiently small, \tilde{r} is still strictly plurisubharmonic near ∂D . Note that the bump B covers a relatively large portion of boundary of D as

$$p \in \partial D \cap B_{\epsilon_2}(p) \subset B \cap \partial D.$$

The $\epsilon_0, \epsilon_1, \epsilon_2$, and δ can be chosen uniformly when p varies on ∂D .

Using the *same* defining function r , we find finitely many p_1, \dots, p_m in ∂D , the associated r_i^* , and the local biholomorphic mappings ψ_i defined on $\omega_i = \mathcal{U}(p_i)$ such that $\psi_i(\omega_i) = B_{\epsilon_0}$, $\psi_i(p_i) = 0$, $\psi_i(B_{\epsilon_2}(p_i)) \subset B_{\epsilon_1}$, while $\{B_{\epsilon_2}(p_i)\}$ is an open covering of ∂D . Set $r_0 = r$ and

$$\hat{N}_i \subset B_{\epsilon_0} : \hat{r}_i(z) := r_i^*(z) + C^* \chi_0(\epsilon_1^{-2} |z|^2) < 0,$$

$$D_{i+1} \subset \mathcal{U}(\overline{D}) : r_{i+1}(z) := r_i(z) - \delta^* \chi_1(\epsilon_2^{-2} |z - p_i|^2) < 0.$$

Set $N_i = \psi_i^{-1}(\hat{N}_i)$ and $B_i = N_i \cup (D_{i+1} \setminus D_i)$. When $\delta^* > 0$ is sufficiently small, $r - r_i$ have small C^2 norms. Thus, we may assume that the ϵ_i, δ have been so chosen that the \hat{r}_i, \hat{N}_i are strictly convex, r_i are strictly plurisubharmonic near ∂D_i , and (3.1) holds. Note that r_i are defined on the domain of r and $r_{i+1} \leq r_i$. Since $\{B_{\epsilon_2}(p_i)\}$ covers ∂D and $B_{\epsilon_2}(p_i) \cap \partial D \subset D_{i+1}$, then $\overline{D} \subset D_m$ as claimed.

We now consider the family $\{\overline{D}^t\}$. Fix t_0 . We apply the above construction to the domain $D = D^{t_0}$. We rename the above $D_i, N_i, B_i, \hat{r}_i, r_i$ by $D_i^{t_0}, N_i^{t_0}, B_i^{t_0}, \hat{r}_i^{t_0}, r_i^{t_0}$, respectively, while the ψ_i is a biholomorphic mapping from ω_i onto B_{ϵ_0} . By Proposition 2.6, we find a family $\{\Gamma^t\}$ of embeddings from \overline{D} onto \overline{D}^t , where t is defined on \overline{I} and I is a neighborhood of t_0 in $[0, 1]$. By the parametrized version of Seeley extension (Lemma 2.3), we may assume that $\{\Gamma^t\} \in C^{a+2,j}(\overline{U} \times \overline{I})$ with Γ^t being extended to embeddings from \overline{U} onto \overline{U}^t . Here $\overline{D} \subset U$. Replacing Γ^t by $\overline{\Gamma^t} \circ (\Gamma^{t_0})^{-1}$, we may assume that Γ^{t_0} is the identity on U . We may also assume that $\overline{D_{m+1}^{t_0}}$ is contained in U^{t_0} . Fix t_0 and define

$$\begin{aligned} D_i^t &= \Gamma^t(D_i), & B_i^t &= \Gamma^t(B_i^{t_0}), & N_i^t &= \Gamma^t(N_i^{t_0}), & \hat{N}_i^t &= \psi_i(N_i^t), \\ \hat{r}_i^t &= \hat{r}_i^{t_0} \circ \psi_i \circ (\Gamma^t)^{-1} \circ (\psi_i)^{-1}, & r_i^t &= r_i^{t_0} \circ (\Gamma^t)^{-1}. \end{aligned}$$

This gives us (ii). We obtain (iii) as follows. When I is sufficiently small, \hat{r}_i^t is a strictly convex defining function of \hat{N}_i^t on $\overline{B_{\epsilon_0}}$ and r_i^t is a defining function of D_i^t on U that is strictly plurisubharmonic near $\overline{U} \setminus D^t$. Also, $\{\hat{r}_i^t\} \in C^{a+2,j}(\overline{B_{\epsilon_0}} \times \overline{I})$ and $\{r_i^t\} \in C^{a+2,j}(\overline{U} \times \overline{I})$. \square

4. BOUNDARY REGULARITY FOR VARIABLE STRICTLY PSEUDOCONVEX DOMAINS

In this section, we study the boundary regularity of the $\bar{\partial}$ -equation on variable strictly pseudoconvex domains. The solutions are obtained first for strictly convex domains. Using a reduction procedure via Grauert's bumps, we then apply the regularity result to the general domains.

Let us start with a homotopy formula constructed by Lieb-Range [20]. Let D be a bounded convex domain with C^{a+2} boundary with $a \geq 0$. Then D has a defining function $r \in C^{a+2}(\mathcal{U}(\partial D))$ that is convex near ∂D . In fact, the signed distance function $\delta_{\partial D}$ is of class C^{a+2} near ∂D (see [5]), and it is convex in \mathbf{C}^n because $\delta_{\partial D}(x) = \sup_{D \subset H} \delta_H(x)$, where H are affine half-spaces in \mathbf{C}^n bounded by hyperplanes. The convexity of D implies that

$$\operatorname{Re}\{r_\zeta \cdot (\zeta - z)\} > 0, \quad \forall \zeta \in \partial D, z \in D.$$

(Recall that in our convention, a convex set is connected.) Let $g^0(\zeta, z) = \bar{\zeta} - \bar{z}$, $g^1(\zeta, z) = r_\zeta$, and $w = \zeta - z$. Define

$$\begin{aligned} \omega^\ell &= \frac{1}{2\pi i} \frac{g^\ell \cdot dw}{g^\ell \cdot w}, & \Omega^\ell &= \omega^\ell \wedge (\bar{\partial} \omega^\ell)^{n-1}, \\ \Omega^{01} &= \omega^0 \wedge \omega^1 \wedge \sum_{\alpha+\beta=n-2} (\bar{\partial} \omega^0)^\alpha \wedge (\bar{\partial} \omega^1)^\beta. \end{aligned}$$

Here Ω^{01} is $\omega^0 \wedge \omega^1$ when $n = 2$ and it is zero for $n = 1$. Note that

$$(4.1) \quad \omega^\ell \wedge (\bar{\partial} \omega^\ell)^\alpha = \frac{g^\ell \cdot dw \wedge (\bar{\partial}(g^\ell \cdot dw))^\alpha}{(2\pi i g^\ell \cdot w)^{\alpha+1}}.$$

Decompose $\Omega^\ell = \sum \Omega_{0,q}^\ell$ and $\Omega^{01} = \sum \Omega_{0,q}^{01}$, where $\Omega_{0,q}^\ell, \Omega_{0,q}^{01}$ are of $(0, q)$ -type in z . We have

$$\bar{\partial}_\zeta \Omega_{0,q}^0 + \bar{\partial}_z \Omega_{0,q-1}^0 = 0, \quad q \geq 1, \quad \bar{\partial}_\zeta \Omega_{0,q}^{01} + \bar{\partial}_z \Omega_{0,q-1}^{01} = \Omega_{0,q}^0 - \Omega_{0,q}^1.$$

We get the homotopy formula for $(0, q)$ form f :

$$(4.2) \quad f(z) = \bar{\partial}_z T_q f + T_{q+1} \bar{\partial}_z f, \quad z \in D, \quad 1 \leq q \leq n,$$

$$(4.3) \quad T_q f = - \int_{\partial D} \Omega_{0,q-1}^{01} \wedge f + \int_D \Omega_{0,q-1}^0 \wedge f, \quad q \geq 1,$$

$$(4.4) \quad f(z) = \int_{\partial D} f(\zeta) \Omega_{0,0}^1(\zeta, z), \quad \forall z \in D, f \in C^1(\bar{D}) \cap \mathcal{A}(D).$$

(See [2, p. 273].)

The formulas (4.2)-(4.4) are valid, provided that r_ζ can be replaced by a *Leray map* $g^1(\zeta, z)$, i.e. it is holomorphic in $z \in D$ and

$$(4.5) \quad g^1(\zeta, z) \cdot (\zeta - z) \neq 0, \quad \forall \zeta \in \partial D, z \in D.$$

Note that r_ζ is never a Leray map when D is not connected. A Leray map g^1 always exists when D has strictly pseudoconvex C^2 boundary. In the latter case there is another homotopy formula constructed via (4.5), where $T_q f$, restricted to a component \tilde{D} of D , is defined by (4.3) in which D is replaced by \tilde{D} . Furthermore, for such a homotopy formula, one only needs a mapping g^1 satisfying

$$g^1(\zeta, z) \cdot (\zeta - z) \neq 0, \quad \forall \zeta \in \partial \tilde{D}, z \in \tilde{D}$$

for each component \tilde{D} of D .

Remark 4.1. With a Leray mapping satisfying (4.5), we have the Leray formula

$$f(z) = \int_{\partial D} f(\zeta) \Omega_{0,0}^1(\zeta, z) = \int_{\partial \tilde{D}} f(\zeta) \Omega_{0,0}^1(\zeta, z), \quad \forall z \in \tilde{D},$$

for $f \in C^1(\bar{D}) \cap \mathcal{O}(D)$. However, the first integral representation is more convenient in holomorphic approximation.

Note that the classical solution operator T_q can be estimated for $\bar{\partial}$ -closed $(0, 1)$ forms; see Siu [26]. For $(0, q)$ forms we recall a $\bar{\partial}$ -solution operator $S_q f$ due to Lieb-Range [20]. We reformulate the Lieb-Range solution operator in terms of the Leray-Koppelman forms for a convex domain.

Recall that the Seeley extension Ef for a differential form f on \bar{D} is obtained by applying E to the coefficients of the form.

Proposition 4.2. *Let $1 \leq q \leq n$. Let $D \subset \mathbf{C}^n$ be a bounded convex domain with C^{a+2} boundary. Let $r \in C^{a+2}(\mathcal{U}(\partial D))$ be a defining function of D . Suppose that $f \in C_{(0,q)}^1(\bar{D})$ is $\bar{\partial}$ -closed on \bar{D} . Let Ef be a C^1 Seeley extension of f that has compact support in $\mathcal{U}(\bar{D})$. Then $\bar{\partial} S_q f = f$ on D for*

$$(4.6) \quad S_q f = L_q Ef + K_q \bar{\partial} Ef,$$

$$(4.7) \quad L_q Ef = \int_{\mathcal{U}(\bar{D})} \Omega_{0,q-1}^0 \wedge Ef, \quad K_q \bar{\partial} Ef = \int_{\mathcal{U}(\bar{D}) \setminus D} \Omega_{0,q-1}^{01} \wedge \bar{\partial}_\zeta Ef.$$

Proof. Let us modify the solution operator T_q given by (4.2)-(4.3). The Ω^{01} has total degree $2n - 2$. Since Ef has compact support in $\mathcal{U}(\overline{D})$, we apply Stokes' formula and get

$$\begin{aligned} - \int_{\partial D} \Omega_{0,q-1}^{01} \wedge f &= \int_{\mathcal{U}(\overline{D}) \setminus D} \overline{\partial}_\zeta \Omega_{0,q-1}^{01} \wedge f + \int_{\mathcal{U}(\overline{D}) \setminus D} \Omega_{0,q-1}^{01} \wedge \overline{\partial} Ef \\ &= - \int_{\mathcal{U}(\overline{D}) \setminus D} (\overline{\partial}_z \Omega_{0,q-2}^{01} \wedge f - \Omega_{0,q-1}^0 \wedge f + \Omega_{0,q-1}^1 \wedge f) + \int_{\mathcal{U}(\overline{D}) \setminus D} \Omega_{0,q-1}^{01} \wedge \overline{\partial} Ef \\ &= - \overline{\partial}_z \int_{\mathcal{U}(\overline{D}) \setminus D} \Omega_{0,q-2}^{01} \wedge f + \int_{\mathcal{U}(\overline{D}) \setminus D} (\Omega_{0,q-1}^0 \wedge f - \Omega_{0,q-1}^1 \wedge f + \Omega_{0,q-1}^{01} \wedge \overline{\partial} Ef). \end{aligned}$$

Let us look at the 4 integrals after the last equal sign. To modify the solution operator, we remove the first integral as $\overline{\partial}^2 = 0$. The third integral of the 4 terms is 0 when $q > 1$, or holomorphic when $q = 1$. In the latter case, we remove it. We end up with two integrals that do not involve boundary integrals. Moreover, the second integral, after combined with the last integral in (4.3), is over the domain $D \cup (\mathcal{U}(\overline{D}) \setminus D) = \mathcal{U}(\overline{D})$. We have verified (4.6). \square

We now apply $S_q f$ to our first case where D^t are strictly convex. More precisely, we assume that $\overline{D^t}$ are strictly convex and have defining functions r^t on U^t which are strictly convex near $\overline{U^t} \setminus D^t$, while $\{r^t\} \in C^{2,0}\{\overline{U^t}\}$. We replace the above $D, \mathcal{U}(\overline{D}), r, E, g^1$ by $D^t, U^t, r^t, E^t, \frac{\partial r^t}{\partial \zeta}$ respectively. Let $S_q^t f$ be the operator S_q applied to $(0, q)$ form f^t on D^t . Thus, we have

$$(4.8) \quad S_q^t f = L_q^t E f + K_q^t \overline{\partial} E f,$$

where $L_q^t E f, K_q^t \overline{\partial} E f$ are defined by (4.7) in which $E f, \mathcal{U}(\overline{D})$ are replaced by $E^t f, \overline{U^t}$ respectively.

In real coordinates x, ξ , we will write $z_j = x_j + i x_{n+j}$ and $\zeta_j = \xi_j + i \xi_{n+j}$. Recall that $\hat{\partial}_\xi^i r^t$ denotes the set of derivatives of $r^t(\zeta)$ of order at most i . In view of (4.1), we can express the coefficients of $S_q^t f$ as follows.

Proposition 4.3. *Let L_q^t, K_q^t be given by (4.7) and (4.8). The coefficients of the $(0, q-1)$ form $K_q^t \overline{\partial} E f(z)$ are \mathbf{C} -linear combinations of*

$$(4.9) \quad K^t f_1(z) := \int_{\xi \in U^t \setminus D^t} f_1^t(\xi) \frac{A(\hat{\partial}_\xi^2 r^t, \xi, x)(\xi_i - x_i)}{(r_\zeta^t \cdot (z - \zeta))^{n-m} |\zeta - z|^{2m}} dV(\xi), \quad \forall x \in D^t$$

with $m = 1, \dots, n-1$ and $i = 1, \dots, 2n$, where f_1^t is a coefficient of $\overline{\partial} E^t f$, A is a polynomial, and dV is the standard volume-form on \mathbf{C}^n . The coefficients of $(0, q-1)$ form $L_q^t f(z)$ are \mathbf{C} -linear combinations of

$$(4.10) \quad L^t f_0(z) := \int_{\xi \in U^t} f_0^t(\xi) \frac{\xi_i - x_i}{|\zeta - z|^{2n}} dV(\xi), \quad 1 \leq i \leq 2n,$$

where f_0^t are coefficients of $E^t f$. The coefficients for the \mathbf{C} -linear combinations are universal and independent of t .

By the proposition, it suffices to show that $\{K^t f_1\}, \{L^t f_0\}$ are in $C^{k+1/2,j}(\overline{D})$, when $\{f_1^t\}, \{f_0^t\}$ are in $C^{k,j}(\{\overline{U^t}\})$.

We first state the following interior estimate. It is valid for $C_*^{k+\alpha,j}$ norm for integers k, j . See [27] for fixed domains.

Proposition 4.4. *Let $j, k \in \mathbf{N}$ and $0 < \alpha < 1$. Let $\{U^t\}$ be a smooth family of domains of $C_*^{k+1+\alpha,j}$ boundary. Let $\{f^t\} \in C_*^{k+\alpha,j}(\{\overline{U^t}\})$ and let $\{L^t f\}$ be defined by (4.10). If \mathcal{D} is a relatively compact open subset of the total space \mathcal{U} of $\{U^t\}$, then for $s_k \leq Ck$,*

$$|Lf|_{\mathcal{D};k+1+\alpha,j} \leq C \operatorname{dist}(\mathcal{D}, \partial\mathcal{U})^{-s_k} |f|_{\mathcal{U};k+\alpha,j}.$$

Proof. This follows directly from the classical estimates for the Newtonian potential. Since \mathcal{D} is relatively compact in \mathcal{U} , we find a smooth function $\chi^t(x)$ in t, x such that $\chi^t f^t$ has compact support in \mathcal{U} , while $\chi^t(x)$ equals 1 near $\overline{\mathcal{D}}$. We may assume that $f^t = \chi^t f^t$ and for $\{f^t\} \in C_*^{k+\alpha,j}$ we have

$$\partial_t^j \{L^t f(z)\} = \int_{B_R} \partial_t^j \{f^t(\xi)\} \frac{\xi_i - x_i}{|\zeta - z|^{2n}} dV(\xi), \quad z \in \overline{D^t} \subset B_R.$$

The estimate in x derivatives is then classical. For detail, see [5] (pp.54-59). \square

When $n = 1$, we have $S_1 f = L_1 E f$ in (4.6). Thus we have proved the following 1-dimensional result.

Theorem 4.5. *Let $j, k \in \overline{\mathbf{N}}$ with $k \geq j$. Let $\{D^t\}$ be a smooth family of non-empty bounded domains in \mathbf{C} with $C^{k+1+\alpha,j}$ boundary. Assume that f^t are $(0, 1)$ forms on D^t with $f = \{f^t\} \in C^{k+\alpha,j}(\overline{\mathcal{D}})$. Then exists linear solution operators $u^t = S^t f$ to $\overline{\partial} u^t = f^t$ on D^t so that $\{S^t f\} \in C^{k+1+\alpha,j}(\overline{\mathcal{D}})$.*

We now estimate the boundary integral to gain $\frac{1}{2}$ in the Hölder exponent.

Proposition 4.6. *Let $j, k \in \mathbf{N}$ with $j < k$. Let $\{U^t\}$ be a smooth family of bounded domains with $C^{k+1,j}$ boundary. Suppose that D^t is non empty and relatively compact in U^t and it is defined by $r^t < 0$ on U^t with $dr^t \neq 0$ on ∂D^t . Suppose that the real Hessian of r^t is strictly positive-definite on $\overline{U^t} \setminus D^t$ and $\{r^t\}$ is of class $C^{k+1,j}(\{\overline{U^t}\})$. Let $\{K^t g\}$ be defined by (4.9). If $g \in C^{k-1,j}(\{\overline{U^t}\})$, $g^t \equiv 0$ on D^t , then*

$$\|Kg\|_{\mathcal{D};k+1/2,j} \leq C \|g\|_{\mathcal{U};k-1,j}.$$

where \mathcal{D}, \mathcal{U} are respectively the total spaces of $\{D^t\}, \{U^t\}$.

Proof. By Proposition 2.6 we know that $\{D^t\}$ is a family of bounded domains of $C^{k+1,j}$ boundary. Using a cut-off function we may assume that each g^t has a compact support in a small neighborhood of ∂D^t . Thus the integral $K^t g$ is over a fixed bounded domain, which simplifies the computation of t derivatives.

Note that $r_\zeta^t \cdot (\zeta - z) \neq 0$, for $z \in D^t$ and $\zeta \in U^t \setminus D^t$. The latter contains the support of g^t . First, we will take the derivatives on the integrand directly. We denote by $N_\nu(x)$ a monomial of degree ν in x . Let $A(w)$ denote a polynomial in w . Also, the A might be different when it recurs. Let us write

$$K^t g(z) := \int_{\mathbf{C}^n} g^t(\xi) \frac{A(\hat{\partial}_\xi^2 r^t, \xi, x) N_1(\xi - x)}{(r_\zeta^t \cdot (\zeta - z))^{n-\ell} |\zeta - z|^{2\ell}} dV(\xi), \quad z \in D^t.$$

Note that $\{g^t\}$ is only in $C^{k-1,j}$. We first compute $\partial_t^i \partial_x^{k_1} \{K^t g(z)\}$ for $i \leq j$ and $i + k_1 < k$. We then apply the integration by parts to derive a new formula. Finally we compute two more derivatives to derive the $\frac{1}{2}$ -estimate by using the Hardy-Littlewood lemma.

We write $\partial_x^{k_1} \{K^t g(z)\}$ as a linear combination of $K_1^t g(x)$ with

$$(4.11) \quad K_1^t g(x) := \int_{\mathbf{C}^n} g^t(\xi) \frac{A(\hat{\partial}_\xi^2 r^t, \xi, x) N_{1-\mu_0+\mu_2}(\xi-x)}{(r_\zeta^t \cdot (\zeta-z))^{n-\ell+\mu_1} |\zeta-z|^{2\ell+2\mu_2}} dV(\xi),$$

$$\mu_0 + \mu_1 + \mu_2 \leq k_1, \quad 1 - \mu_0 + \mu_2 \geq 0.$$

It suffices to estimate $\partial_t^i \{K_1^t g(x)\}$, which is a linear combination of

$$(4.12) \quad J^t(x) := \int \frac{\hat{g}^t(\xi) N_{1-\mu_0+\mu_2+j_2}(\xi-x)}{(r_\zeta^t \cdot (\zeta-z))^{n-\ell+\mu_1+j_2} |\zeta-z|^{2\ell+2\mu_2}} dV(\xi)$$

with

$$(4.13) \quad \hat{g}^t(\xi) = \partial_t^{j_0}(g^t(\xi)) A(\partial_t^{j_1} \hat{\partial}_\xi^2 r^t, \partial_t r_\zeta^t, \xi, x).$$

Furthermore, μ_0, μ_1, μ_2 satisfying (4.11) and

$$j_0 + j_1 + j_2 \leq i, \quad i \leq j, \quad i + k_1 \leq k - 1.$$

Next, we will apply the integration by parts to reduce the exponent of $r_\zeta^t \cdot (\zeta-z)$ to $n-\ell$. This requires us to transport the derivative in t to derivatives in ξ . Thus, we need the space C^\bullet instead of C_\bullet^\bullet . To this end we write (4.12) as

$$(4.14) \quad J_m^t(x) = \int_{\mathbf{C}^n} \frac{\tilde{g}^t(\xi, x)}{(r_\zeta^t \cdot (\zeta-z))^{n-\ell+m}} dV(\xi), \quad m = \mu_1 + j_2,$$

$$(4.15) \quad \tilde{g}^t(\xi, x) = \hat{g}^t(\xi) \frac{N_{1-\mu_0+\mu_2+j_2}(\xi-x)}{|\zeta-z|^{2\ell+2\mu_2}}.$$

Using a partition of unity in (ζ, t) space, we may assume that $g^t(\zeta)$ has compact support in a small ball B centered at (ζ_0, t_0) , and on B

$$\partial_\zeta \{r_\zeta^t \cdot (\zeta-z)\} \neq 0, \quad u^t(\xi, x) := \partial_{\xi_\beta} \{r_\zeta^t \cdot (\zeta-z)\} \neq 0,$$

for some β . Since $g^t(\zeta)$ is 0 on D^t and has a compact support in B , we apply Stokes' theorem and obtain that, up to a constant multiple

$$J_m^t(x) = \int_{\mathbf{C}^n} \frac{\partial_{\xi_\beta} \{u^t(\xi, x)^{-1} \tilde{g}^t(\xi, x)\}}{(r_\zeta^t \cdot (\zeta-z))^{n-\ell+m-1}} dV(\xi), \quad \forall z \in D^t.$$

Repeating this shows that up to a constant

$$J_m^t(x) = \int_{\mathbf{C}^n} \frac{v^t(\xi, x)}{(r_\zeta^t \cdot (\zeta-z))^{n-\ell}} dV(\xi), \quad \forall z \in D^t$$

with

$$(4.16) \quad v^t(\xi, x) := (\partial_{\xi_\beta} \circ u^t(\xi, x)^{-1})^m \{\tilde{g}^t(\xi, x)\}.$$

Since $\tilde{g}^t(\xi, x) = 0$ for $\xi \in D^t$, it is easy to see that $J_m^t(x)$, $\partial_x J_m^t$ are continuous on \mathcal{D} . To show that $\{K^t g\}$ is in $C^{k+1/2,j}(\{\overline{D^t}\})$, by Hardy-Littlewood lemma it suffices to verify

$$|\partial_x^2 J_m^t(x)| \leq C \operatorname{dist}(x, \partial D^t)^{-1/2}, \quad \forall x \in D^t.$$

Since $g^t(x) = 0$ for $x \in D^t$, we obtain that for $b \leq j, a + b \leq k - 1$

$$|\partial_{\xi_\beta}^a \partial_t^b \{g^t(\xi)\}| = |\partial_{\xi_\beta}^a \partial_t^b \{g^t(\xi)\} - \partial_{x_\beta}^a \partial_t^b \{g^t(x)\}| \leq C \|g\|_{\mathcal{U}; k-1, j} |\xi - x|^{k-1-a-b}.$$

Thus, $|g^t(\xi)| \leq C |\xi - x|^{k-1}$, $|\hat{g}^t(\zeta)| \leq C |\xi - x|^{k-1-j_0}$ by (4.13), and $|\tilde{g}(\xi, x)| \leq C |\xi - x|^\nu$ by (4.15), for

$$\nu = (k - 1 - j_0) + (1 - \mu_0 + \mu_2 + j_2) - (2\ell + 2\mu_2).$$

By (4.16), we get

$$|v^t(\xi, x)| \leq C (\|r\|_{\mathcal{U}; k+1, j}) \|g\|_{\mathcal{D}; k-1, j} |\xi - x|^{\nu'}, \quad \nu' = \nu - m.$$

Recall from (4.14) that $m = \mu_1 + j_2$. Hence

$$\begin{aligned} \nu' &= (k - 1 - j_0) + (1 - \mu_0 + \mu_2 + j_2) - (2\ell + 2\mu_2) - (\mu_1 + j_2) \\ &= k - j_0 - \mu_0 - 2\ell - \mu_2 - \mu_1 \geq k - i - k_1 - 2\ell \geq 1 - 2\ell. \end{aligned}$$

We obtain similar estimates for $|\partial_x^s v^t(\xi, x)|$. In summary, we have

$$|\partial_x^s v^t(\xi, x)| \leq C \|g\|_{\mathcal{D}; k-1, j} |\xi - x|^{(1-2\ell)-s}, \quad s = 0, 1, 2.$$

Here C depends on $\|r\|_{\mathcal{U}; k+1, j}$. Now, $\partial_x^2 J_m(x)$ is a linear combination of

$$\begin{aligned} J_{m,0}^t(x) &:= \int_{\mathbb{C}^n} \frac{\partial_x^2 \{v^t(\xi, x)\}}{(r_\zeta^t \cdot (\zeta - z))^{n-\ell}} dV(\xi), \\ J_{m,1}^t(x) &:= \int_{\mathbb{C}^n} \frac{A(r_\zeta^t, \xi, x) \partial_x \{v^t(\xi, x)\}}{(r_\zeta^t \cdot (\zeta - z))^{n-\ell+1}} dV(\xi), \\ J_{m,2}^t(x) &:= \int_{\mathbb{C}^n} \frac{A(r_\zeta^t, \xi, x) v^t(\xi, x)}{(r_\zeta^t \cdot (\zeta - z))^{n-\ell+2}} dV(\xi), \end{aligned}$$

for $x \in D^t$. Therefore, we obtain

$$(4.17) \quad |J_{m,i}^t(x)| \leq C \int_{U^t \setminus D^t} \frac{dV(\xi)}{|r_\zeta^t \cdot (\zeta - z)|^{n-\ell+i} |\zeta - z|^{2\ell+1-i}}.$$

Since $|r_\zeta^t \cdot (\zeta - z)| \geq C |\zeta - z|^2$, it suffices to estimate (4.17) for $\ell = n - 1$. We have

$$|J_{m,i}^t(x)| \leq C \int_{U^t \setminus D^t} \frac{1}{|r_\zeta^t \cdot (\zeta - z)|^{i+1} |\zeta - z|^{2n-i-1}} dV(\xi), \quad i = 0, 1, 2.$$

Then the last integrals are bounded by $C \text{dist}(z, \partial D^t)^{-1/2}$. For the further detail, see Lieb-Range (the estimates of $J_k(z)$ in [20], pp. 155–166.) \square

To study regularity of $\bar{\partial}$ -solutions for variable domains, we need to introduce the following.

Definition 4.7. A family $\{E^t\}$ of subsets in a topological space X is *upper semi-continuous*, if for each t and every open neighborhood U of E^t in X , we have $E^s \subset U$ when $|s - t|$ is sufficiently small.

It is easy to see that the family $\{X \setminus E^t\}$ might not be upper semi-continuous in X when $\{E^t\}$ is upper semi-continuous in X .

Remark 4.8. The family of boundaries of domains defined by $r^t < c$ is not necessarily upper semi-continuous.

Lemma 4.9. *A family $\{K^t\}$ of compact sets K^t in \mathbf{R}^d is upper semi-continuous if and only if its total space \mathcal{K} is compact. In particular, if $\{K^t\}$ has a compact total space \mathcal{K} and the total space of $\{\omega^t\}$ is an open subset of \mathcal{K} , then $\{K^t \setminus \omega^t\}$ is upper semi-continuous.*

Proof. Obviously, the last assertion follows from the first assertion. Suppose that $\{K^t\}$ is upper-semi continuous with total space \mathcal{K} . If \mathcal{K} is not compact, there is a sequence $(x_m, t_m) \in \mathcal{K}$ that does not admit any convergent subsequence with limit in \mathcal{K} . We may assume that $t_m \rightarrow t_0$ as $m \rightarrow \infty$. Since K_{t_0} is bounded, it is contained in an open ball U of finite radius. By the upper-semi continuity, we know that $K_{t_m} \subset U$ for m sufficiently large. We may assume that $x_m \rightarrow x_0$. Then x_0 is not in K_{t_0} . Take another open set U' containing K_{t_0} such that x_0 is not in $\overline{U'}$. By the upper semi-continuity, we have $K_{t_m} \subset U'$. Then x_0 is in $\overline{U'}$, a contradiction.

Suppose now that \mathcal{K} is compact. Fix $t_0 \in [0, 1]$. Let U be an open neighborhood of K^{t_0} in \mathbf{R}^d . Suppose that K^{t_m} is not contained in U for a sequence $t_m \rightarrow t_0$. Take $x_m \in K^{t_m} \setminus U$. Since \mathcal{K} is compact, taking a subsequence if necessary we conclude that (x_m, t_m) tends to $(x_0, t_0) \in \mathcal{K}$. This shows that $x_0 \in K^{t_0} \setminus U$ and the latter is non-empty, a contradiction. \square

Theorem 4.10. *Let $j, k \in \overline{\mathbf{N}}$ with $k - 1 \geq j$. Let $\{D^t\}$ be a smooth family of non-empty bounded domains in \mathbf{C}^n with $C^{k+1,j}$ boundary. Assume that D^t are strongly pseudoconvex. Assume that f^t are $\bar{\partial}$ -closed $(0, q)$ forms on D^t with $q > 0$ and $f = \{f^t\} \in C^{k,j}(\overline{\mathcal{D}})$. There exist linear solution operators $u^t = S^t f$ to $\bar{\partial} u^t = f^t$ on D^t so that $\{S^t f\} \in C^{k+1/2,j}(\overline{\mathcal{D}}) \cap C^{k+\epsilon,j}(\mathcal{D})$ for all $\epsilon < 1$.*

Proof. We first consider the case when all $\overline{D^t}$ are strictly convex. By Proposition 2.6 we can find defining functions r^t for D^t , where $\{r^t\} \in C^{k+1,j}(\overline{\mathcal{U}})$; and r^t have positive-definite real Hessian on ∂D^t replacing r^t by $e^{Cr^t} - 1$ if necessary. Then we have homotopy formula (4.8) that provides a solutions operators S^t . The regularity follows from Propositions 4.4 and 4.6.

The proof for the general case consists of two steps. We first use the bumps in Lemma 3.1 and the theorem for the strictly convex domains to extend $\{f^t\}$ to a family of $\bar{\partial}$ -closed forms on larger domains. We then solve the $\bar{\partial}$ -equation on a fixed large domain by using the classical homotopy formula. Note that we only constructed bumps uniformly in t for t close to a given value. Thus, we will first define the solution operator S^t locally in t and we will then define S^t for all t by using a partition of unity in parameter t .

We recall the construction from Lemma 3.1. Fix t_0 . We can find a connected neighborhood I of t_0 such that when restricting t to \overline{I} we have the following: there are finitely many strictly pseudoconvex domains D_i^t, B_i^t, N_i^t with C^{k+2} boundary such that

$$\begin{aligned} D_{i+1}^t &= D_i^t \cup B_i^t, & \overline{D^t} &\subset D_m^t, & N_i^t &= D_i^t \cap B_i^t, & D_0^t &= D^t; \\ (4.18) \quad & & (\overline{B_i^t} \setminus \overline{D_i^t}) \cap (\overline{D_i^t} \setminus \overline{N_i^t}) &= \emptyset, \end{aligned}$$

$$(4.19) \quad \Gamma^t(D_i) = D_i^t, \quad \Gamma^t(N_i) = N_i^t, \quad \Gamma_i^t(B_i) = B_i^t$$

and there exists a biholomorphic mapping ϕ_i from ω_i onto B_ϵ , independent of t , such that $\hat{N}_i^t := \psi_i(N_i^t)$ is strictly convex. Furthermore, B_i, N_i and D_i are relatively compact in U

and $\{\Gamma^t\} \in C^{k+1,j}(\overline{U})$, and the D_i^t (resp. \hat{N}_i^t) is defined by $r_i^t < 0$ (resp. $\hat{r}_i^t < 0$). The $\{r_i^t\}$ is in $C^{k+1,j}(\overline{U})$ and \hat{r}_i^t is strictly convex on $\overline{B_\epsilon}$. Here U contains $\overline{D^t}$ and has C^{k+1} boundary.

Let $S_{\hat{N}_i^t}$ be the Lieb-Range solution operator determined by \hat{r}_i^t for \hat{N}_i^t . We pull back the solution operator to N_i^t and define $S_{N_i^t}^t g := (\psi_i)^* S_{\hat{N}_i^t}(\psi_i^{-1})^* g^t$.

By (4.19) we know that

$$(\overline{B_i^t \setminus D_i^t}) \cap \partial D_i^t = \Gamma^t((\overline{B_i \setminus D_i}) \cap \partial D_i), \quad \overline{D_i^t \setminus N_i^t} = \Gamma^t(\overline{D_i \setminus N_i})$$

are upper semi-continuous in \mathbf{C}^n . Thus we can find open neighborhoods U_i^0, U_i^1 of $(\overline{B_i^t \setminus D_i^t})$ such that $\overline{U_i^1} \subset U_i^0$ and $\overline{U_i^0} \cap (\overline{D_i^t \setminus N_i^t}) = \emptyset$ for t near t_0 . Now we find a smooth function χ_i that has compact support in U_i^0 such that $\chi_i = 1$ near U_i^1 . Since we have only finitely many families $\{N_i^t\}$, the above construction of $S_{N_i^t}^t$ is valid for all t near t_0 and all i . We then define

$$f_1^t = f^t - \overline{\partial}(\chi_0 S_{N_0}^t f) = (1 - \chi_0)f^t - (\overline{\partial}\chi_0) \wedge S_{N_0}^t f.$$

The last identity implies that f_1^t vanishes near $(\overline{B_0^t \setminus D_0^t})$. We extend f_1^t to be zero on $\overline{D_1^t \setminus D_0^t}$. Then f_1^t is $\overline{\partial}$ -closed on D_1^t , and $\{f_1^t\} \in C^{k,j}(\{\overline{D_1^t}\})$. We define $f_{i+1}^t = f_i^t - \overline{\partial}(\chi_i S_{N_i}^t f_i)$ and extend it to zero on $\overline{D_{i+1}^t \setminus D_i^t}$. We can write

$$f_m^t = f^t - \overline{\partial}g^t, \quad g^t = \sum_{i=0}^{m-1} \chi_i S_{N_i}^t f_i.$$

We have $\{g^t\} \in C^{k+1/2,j}(\overline{\mathcal{D}})$, while $\{f_m^t\} \in C^{k,j}(\{\overline{D_{m+1}^t}\})$. Since χ_i is contained in N_i^t , then $\{\chi_i S_{N_i}^t f_i\} \in C^{k+\epsilon,j}(\mathcal{D})$ for all $\epsilon < 1$. This shows that $\{g^t\}$ is in $C^{k+1/2,j}(\overline{\mathcal{D}}) \cap C^{k+\epsilon,j}(\mathcal{D})$.

Again, for the fixed t_0 , we can find a strictly pseudoconvex domain D_* of C^2 boundary such that $\overline{D^t} \subset D_*$ for t near t_0 . Let T be the solution operator from the classical homotopy formula on D_* . By the interior regularity property of T_{D_*} , we get $\{T_{D_*} f_m^t\} \in C^{k+\epsilon}(\overline{D_\Gamma})$. Then $S^t f := T_{D_*} f_m^t + g^t$ is a solution operator of the desired property for t near t_0 . Using a partition of unity $\{\tilde{\chi}_i\}$ on $[0, 1]$ with $\text{supp } \tilde{\chi}_i \subset (a_i, b_i)$, we obtain a solution operator $\sum \tilde{\chi}_i(t) S_i^t$, where S_i^t is defined for $t \in (a_i, b_i)$. Then $S^t := \sum \tilde{\chi}_i(t) S_i^t$ has the desired properties. \square

5. HENKIN-RAMÍREZ FUNCTIONS FOR VARIABLE STRICTLY PSEUDOCONVEX OPEN SETS

In this section we will construct a family of Henkin-Ramírez functions for variable strictly pseudoconvex open sets. As an application, we will find homotopy formulas for a smooth family $\{D^t\}$ of strictly pseudoconvex domains.

The following theorem is on Henkin-Ramírez functions with parameter.

Theorem 5.1. *Let $a, b \in \overline{\mathbf{R}}_+$, $j \in \overline{\mathbf{N}}$, and $j \leq a$. Let $\{\omega^t\}, \{D^t\}, \{U^t\}$ respectively be continuous families of domains with total spaces $\omega, \mathcal{D}, \mathcal{U}$. Suppose that ω is relatively compact in \mathcal{U} . Let $\{r^t\}$ be of class $C^{a+2,j}(\mathcal{U})$ (resp. $C_*^{b+2,j}(\mathcal{U})$). Suppose that r^t are strictly plurisubharmonic on ω^t . Let $C^t := \omega^t \cap \{r^t = 0\}$. Suppose that for each $t \in [0, 1]$*

$$(5.1) \quad C^t \neq \emptyset, \quad \partial D^t \subset C^t \subset \subset \omega^t,$$

$$(5.2) \quad r^t < 0 \quad \text{on } D^t, \quad r^t > \delta_0 \quad \text{on } U^t \setminus (D^t \cup \omega^t)$$

with $\delta_0 > 0$. For $\delta > 0$, set $D_\delta^t := D^t \cup \{z \in \omega^t : r^t(z) < \delta\}$, $D_{-\delta}^t := \{z \in D^t : r^t(z) < -\delta\}$, and $V_\delta^t := \{z \in D^t \cup \omega^t : |r^t(z)| < \delta\}$. Assume that

$$(5.3) \quad \sum \frac{\partial^2 r^t}{\partial \zeta_j \partial \bar{\zeta}_k} s_j \bar{s}_k \geq \lambda_0 |s|^2, \quad \forall \zeta \in \overline{\omega^t},$$

$$(5.4) \quad |\partial_\zeta^2 r^t - \partial_z^2 r^t| < \frac{\lambda_0}{C_n}, \quad \forall \zeta \in V_{\delta_0}^t, z \in \omega^t \cup D^t, |\zeta - z| < d_0$$

for some $\lambda_0 > 0$ and $d_0 > 0$. Let $0 < \delta_1 < \delta_0$ and

$$(5.5) \quad d = \min \left\{ d_0, \text{dist}(V_{\delta_1}^t, \partial \omega^t \setminus D^t) : t \in [0, 1] \right\}, \quad \epsilon = \min \left\{ \frac{\lambda_0}{64} d^2, \delta_1 \right\}.$$

Then $d > 0$ and there exist functions $\Phi^t(z, \zeta)$ and

$$F^t(z, \zeta) = - \sum \frac{\partial r^t}{\partial \zeta_j} (z_j - \zeta_j) - \frac{1}{2} \sum b_{jk}^t(\zeta) (z_j - \zeta_j) (z_k - \zeta_k)$$

so that for $z \in D_{\epsilon/2}^t$ and $\zeta \in D_{\delta_1}^t \setminus D_{-\epsilon}^t$ the following hold:

- (i) The functions $\Phi^t(z, \zeta)$ are holomorphic in z , and $\Phi^t(z, \zeta) \neq 0$ for $z \neq \zeta$ and $r^t(z) \leq r^t(\zeta)$.
- (ii) If $|\zeta - z| < \epsilon$, there exist $M^t(\zeta, z) \neq 0$ such that $\Phi^t(\zeta, z) = F^t(\zeta, z) M^t(\zeta, z)$ and

$$(5.6) \quad \text{Re } F^t(z, \zeta) \geq r^t(\zeta) - r^t(z) + \frac{\lambda_0}{4} |\zeta - z|^2, \quad \text{if } |\zeta - z| < d, \zeta, z \in D_{\delta_1}^t.$$

- (iii) The families $\{\Phi^t\}$, $\{M^t\}$ are in $C^{a+1,j}(\{\overline{D_{\epsilon/2}^t} \times (\overline{D_{\delta_1}^t} \setminus D_{-\epsilon}^t)\})$ (resp. $C_*^{b+1,j}(\{\overline{D_{\epsilon/2}^t} \times (\overline{D_{\delta_1}^t} \setminus D_{-\epsilon}^t)\})$).

Remark 5.2. (i) The main conclusion is about the uniform size ϵ , given in (5.5), of the band $D_{\epsilon/2}^t \times (D_{\delta_1}^t \setminus D_{-\epsilon}^t)$ on which $\Phi^t(\zeta, z)$ are defined. This will be crucial in proving a parametrized version of Oka-Weil approximation in section 6. (ii) ∂D^t might not be smooth and D^t might not be connected. Furthermore, D^t could be empty when C^t consists of local minimum points of φ^t . (iii) The results are classical for non-parameter case. When t is fixed and V_δ^t is replaced by a neighborhood of ∂D^t , and r^t is of class C^2 , see [12, Theorem 2.4.3, p. 78; Theorem 2.5.5, p. 81]. For the case when ∂D^t has finitely smooth boundary, see Range [24, Proposition 3.1, p. 284].

Proof. To simplify notation, we first derive some uniform estimates without parameter. We then make necessary adjustments for the parametrized version.

(i) Let us first assume that r^t is independent of t . Write $D^t, V_\delta^t, D_\delta^t$ as D, V_δ, D_δ respectively. We consider the Levi polynomial of r at ζ

$$F_0(z, \zeta) := - \sum \frac{\partial r}{\partial \zeta_j} (z_j - \zeta_j) - \frac{1}{2} \sum \frac{\partial^2 r}{\partial \zeta_j \partial \bar{\zeta}_k} (z_j - \zeta_j) (z_k - \zeta_k).$$

Assume that

$$(5.7) \quad \zeta \in V_{\delta_1}, \quad z \in D_{\delta_1}, \quad |\zeta - z| < d.$$

Let $f(t) = r((1-t)\zeta + tz)$. Then $f(1) - f(0) = f'(0) + \frac{1}{2}f''(0) + \frac{1}{2}(f''(s) - f''(0))$ for some $0 < s < 1$. By (5.4)-(5.5), and (5.7), we get

$$2 \operatorname{Re} F_0(z, \zeta) \geq r(\zeta) - r(z) + \lambda_0 |\zeta - z|^2 - C'_n \max_t |\partial^2 r((1-t)\zeta + tz) - \partial^2 r(\zeta)| |\zeta - z|^2.$$

Therefore, if ζ, z satisfy (5.7), then

$$(5.8) \quad 2 \operatorname{Re} F_0(z, \zeta) \geq r(\zeta) - r(z) + \frac{\lambda_0}{2} |\zeta - z|^2.$$

Using a real smooth function $\chi \geq 0$ with compact support in the unit ball of \mathbf{C}^n such that $\int \chi = 1$, let us define

$$\chi_d(z) = d^{-2n} \chi(d^{-1}z), \quad a_{ij}(z) = \int \frac{\partial^2 r}{\partial \zeta_i \partial \zeta_j} (z - \zeta) \chi_d(\zeta) dV(\zeta).$$

Then we get C^∞ functions a_{ij} such that on $D_{\delta_1} \setminus D_{-\delta_1}$

$$\sup_{\zeta \in \omega} \left| a_{ij}(\zeta) - \frac{\partial^2 r}{\partial \zeta_i \partial \zeta_j} \right| < C'_n \frac{\lambda_0}{C_n},$$

$$|a_{ij}|_a \leq C''_a |r|_{a+2}, \quad |a_{ij}|_{a+1} \leq C''_a d^{-1} |r|_{a+2}.$$

We replace the Levi polynomial F_0 by

$$F(z, \zeta) := - \sum \frac{\partial r}{\partial \zeta_j} (z_j - \zeta_j) - \frac{1}{2} \sum a_{ij}(\zeta) (z_i - \zeta_i) (z_j - \zeta_j).$$

Now (5.8) implies that

$$(5.9) \quad 2 \operatorname{Re} F(z, \zeta) \geq r(\zeta) - r(z) + \frac{\lambda_0}{4} |\zeta - z|^2$$

if ζ and z satisfy (5.7). This shows that if ζ, z satisfy

$$(5.10) \quad d/2 < |\zeta - z| < d, \quad \zeta, z \in D_{\delta_1},$$

$$(5.11) \quad r(\zeta) > r(z) - \frac{\lambda_0 d^2}{32},$$

then we must have

$$(5.12) \quad 2 \operatorname{Re} F(z, \zeta) \geq \frac{\lambda_0}{4} \times \frac{d^2}{4} - \frac{\lambda_0 d^2}{32} = \frac{\lambda_0 d^2}{32}.$$

Let χ be a C^∞ function such that $\chi(\zeta) = 1$ for $|\zeta| < \frac{3d}{4}$ and $\chi(\zeta) = 0$ for $|\zeta| > \frac{7d}{8}$. Define

$$(5.13) \quad f(z, \zeta) = \begin{cases} \bar{\partial}_z (\chi(\zeta - z) \log F(z, \zeta)) & \text{if } d/2 < |\zeta - z| < d, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the coefficients of f are C^{a+1} in ζ, z if

$$z \in D_\epsilon, \quad \zeta \in D_{\delta_1} \setminus D_{-\epsilon}, \quad \epsilon = \min \left\{ \delta_1, \frac{\lambda_0 d^2}{64} \right\}.$$

Indeed, the conditions imply (5.11). If (5.10) holds, then (5.12) implies that f is defined in the first case of (5.13). So it is of class C^{a+1} . If (5.10) fails, then it is defined in the second

case. Furthermore, near $|\zeta - z| = d/2$ or d , (5.11) and (5.12) imply that f is identically zero. Therefore, f is of class C^{a+1} .

We claim that $r \geq \delta_0$ on $\partial\omega \setminus D$ and it takes all values $[0, \delta_0]$ in ω . Indeed, by (5.1), ω is not contained in D . Then $m := \max\{r(z) : z \in \partial\omega\} \geq \delta_0$ by (5.2). For any $c \in [0, \delta_0]$, we find a line segment γ with $\gamma(0) \in C(t)$, $m' := r(\gamma(1)) \in [c, m]$, and $\gamma(1) \in \omega$. Let s' be the largest s such that $r(\gamma(s)) = 0$, and let s'' be the smallest $s \in [s_*, 1]$ such that $r(\gamma(s)) = m'$. Thus $\gamma((s', s'')) \subset \omega \setminus \overline{D}$ and $r(\gamma((s', s''))) = [0, m']$; the claim is verified. By Sard's theorem, the r attains a regular value $\epsilon' \in [4\epsilon/5, 9\epsilon/10]$ on ω . Recall that $D_{\epsilon'} = D \cup \{z \in \omega : r(z) < \epsilon'\}$. By (5.2), we obtain

$$\partial D_{\epsilon'} = \omega \cap \{r = \epsilon'\}$$

which is compact and of class C^{a+2} .

Let $T_{D_{\epsilon'}}$ be a solution operator from the classical homotopy formula for the strictly pseudoconvex domain $D_{\epsilon'}$. (Note that the construction does not require the domain to be connected, as shown in Theorem 5.8 below for the parametrized version.)

Define

$$u(z, \zeta) = T_{D_{\epsilon'}} f(\cdot, \zeta)(z), \quad \forall \zeta \in D_{\delta_1} \setminus D_{-\epsilon}, z \in D_{\epsilon'}.$$

By the interior estimate, we obtain $u \in C^{a+1}(D_{\epsilon'})$ as $f \in C^{a+1}(D_{\epsilon'})$. We also have

$$\frac{\partial^{|\beta|} u(z, \zeta)}{\partial \xi^\beta} = T_{D_{\epsilon'}} \left\{ \frac{\partial^{|\beta|} f(\cdot, \zeta)}{\partial \xi^\beta} \right\} (z),$$

which is continuous on $D_{\epsilon'} \times (D_{\delta_1} \setminus D_{-\epsilon})$ for $|\alpha| + |\beta| \leq a + 1$. Therefore,

$$T_{D_{\epsilon'}} f(\cdot, \zeta)(z) \in C^{a+1}(D_{\epsilon'} \times (D_{\delta_1} \setminus D_{-\epsilon})).$$

By (5.9), we can define $\log F(z, \zeta)$ for $0 < |\zeta - z| < \epsilon$ to define

$$\Phi(z, \zeta) = \begin{cases} F(z, \zeta) e^{-u(z, \zeta)} & \text{if } |\zeta - z| < d, \\ e^{\chi(\zeta - z) \log F(z, \zeta) - u(z, \zeta)} & \text{otherwise.} \end{cases}$$

This shows that $\Phi \in C^{a+1}(D_{\epsilon'} \times (D_{\delta_1} \setminus D_{-\epsilon}))$.

We now consider the parametrized version. Let us first show that d , defined by (5.5), is positive. (Note that we cannot conclude that $\partial\omega^t \setminus D^t$, $\partial V_{\delta_0}^t$ are upper-semi continuous.) Otherwise, there are sequences $z_k \in V_{\delta_1}^{t_k}$, $z'_k \in \partial\omega^{t_k} \setminus D^{t_k}$ with $t_k \rightarrow t_0$ as $k \rightarrow \infty$ such that $|z'_k - z_k| \rightarrow 0$ as $k \rightarrow \infty$. By (5.1), we know that \mathcal{D} is relatively compact in \mathcal{U} as ω is relatively compact in \mathcal{U} . Taking a subsequence, we may assume that $z_k \rightarrow z_0$ and $z'_k \rightarrow z_0$ as $k \rightarrow \infty$. We have $\partial\omega^t \setminus D^t \subset U^t \setminus (D^t \cup \omega^t)$; hence (5.2) implies that $r(z'_k) > \delta_0$. We have $z_k \in V_{\delta_0}^{t_k}$; hence $|r^{t_k}(z_k)| \leq \delta_1$. Letting $k \rightarrow \infty$, we obtain that $r^{t_0}(z_0) \geq \delta_0$ and $r^{t_0}(z_0) \leq \delta_1$, a contradiction.

We first remark that r^t is defined on U^t . For D^t defined by $r^t < 0$ (possibly empty), we take

$$F^t(z, \zeta) := - \sum \frac{\partial r^t}{\partial \zeta_j} (z_j - \zeta_j) - \frac{1}{2} \sum b_{ij}^t(\zeta) (z_i - \zeta_i) (z_j - \zeta_j).$$

Here by Proposition 2.8, we have chosen $\{b_{\alpha\beta}^t\} \in C^{\infty,\infty}$ such that $|b_{\alpha\beta}^t(\zeta) - \frac{\partial^2 r^t}{\partial \zeta_\alpha \partial \bar{\zeta}_\beta}| < \lambda_0/4$. We have (5.6). We then define

$$f^t(z, \zeta) = \begin{cases} \bar{\partial}_z(\chi(\zeta - z) \log F^t(z, \zeta)) & \text{if } d/2 < |\zeta - z| < d, \\ 0 & \text{otherwise.} \end{cases}$$

As before, we can verify that $\{f^t\}$ is of class $C^{a+1,j}(\{D_\epsilon^t \times (D_{\delta_1}^t \setminus \overline{D_{-\epsilon}^t})\})$.

Fix t_0 . We apply the above to $D = D^{t_0}$ and denote $D_{\epsilon'}$ by $D_{\epsilon'}^{t_0}$. Applying Lemma 4.9 to $\phi^t = r^t$, we conclude that $D_{\epsilon/2}^t \subset D_{\epsilon'}^{t_0} \subset D_\epsilon^t$ when t is close to t_0 . Assume that t is sufficiently close to t_0 . Then we obtain

$$(5.14) \quad \begin{aligned} u^t(z, \zeta) &= T_{D_\epsilon^{t_0}} f^t(\cdot, \zeta)(z), \quad \zeta \in D_{\delta_0}^t \setminus D_{-\epsilon}^t, \quad z \in D_{\epsilon/2}^t, \\ \Phi^t(z, \zeta) &= \begin{cases} F^t(z, \zeta) e^{-u^t(z, \zeta)} & \text{if } |\zeta - z| < d, \\ e^{\chi(\zeta - z) \log F^t(z, \zeta) - u^t(z, \zeta)} & \text{otherwise.} \end{cases} \end{aligned}$$

Since $\{r^t\} \in C^{a+2,j}(\{\overline{U^t}\})$, then $\{\partial_\zeta r^t\} \in C^{a+1,j}(\{\overline{U^t}\})$.

We now use a partition $\{\chi_\nu\}$ of unity on $[0, 1]$ such that each χ_ν has compact support in a small neighborhood $\mathcal{U}(t_\nu)$ of t_ν and we define (5.14). We then define

$$\begin{aligned} \tilde{u}^t(z, \zeta) &= \sum \chi_\nu(t) T_{D_\nu^{t_\nu}} f^t(\cdot, \zeta)(z), \\ \tilde{\Phi}^t(z, \zeta) &= \begin{cases} F^t(z, \zeta) e^{-\tilde{u}^t(z, \zeta)} & \text{if } |\zeta - z| < d, \\ e^{\chi(\zeta - z) \log F^t(z, \zeta) - \tilde{u}^t(z, \zeta)} & \text{otherwise.} \end{cases} \end{aligned}$$

For $z \in \mathcal{U}(\overline{D^t})$, we have $\bar{\partial} \tilde{u}^t = \sum_\nu \chi_\nu(t) f^t(z, \zeta) = f^t(z, \zeta)$ for $\bar{\partial}$ in \bar{z} . Define Φ^t , M^t in the theorem to be $\tilde{\Phi}^t$, $e^{-\tilde{u}^t}$ respectively. Note that the integral operator $T_{D_\nu^{t_\nu}}$ is independent of t , we can verify (iii). The proof for $C_*^{b+1,j}$ regularity is almost identical as the solution operator in (5.14) is independent of t . Thus we do not repeat the argument here. \square

We need Oka-Hefer decomposition with multi parameters. Let us first introduce the following.

Definition 5.3. Let $a, b \in \overline{\mathbf{R}}_+$ and let $j \in \overline{\mathbf{N}}$ with $j \leq a$. Let $\{w^t\}, \{Y^t\}$ be continuous families of domains in $\mathbf{C}^n, \mathbf{R}^d$, respectively. Denote by $C^{a,j}(\{Y^t, \mathcal{O}(\omega^t)\})$ the subspace of $\{f^t\} \in C^{a,j}(\{Y^t \times \omega^t\})$ satisfying $f^t(y, \cdot) \in \mathcal{O}(\omega^t)$ for all $y \in Y^t$. We define $C_*^{b,j}(\{Y^t, \mathcal{O}(\omega^t)\})$ analogously.

Lemma 5.4. Let $a \in \overline{\mathbf{R}}_+$ and let $j \in \overline{\mathbf{N}}$ with $j \leq a$. Let $\{D^t\}$ be a continuous family of domains in \mathbf{C}^n with total space \mathcal{D} . Let $\{\varphi^t\} \in C^{0,0}(\mathcal{D})$ be a family of plurisubharmonic functions φ^t on D^t . Let

$$\omega_c^t = \{z \in D^t : \varphi^t(z) < c\}$$

with total space ω_c . Let $0 < c_1 < c_0$. Assume that ω_{c_0} is relatively compact in \mathcal{D} . Let H be a complex hyperplane in \mathbf{C}^n . Assume that $H \cap \omega_{c_1}^t \neq \emptyset$, for all $t \in [0, 1]$. There exists a linear continuous extension mapping

$$E: C^{a,j}(\{Y^t, \mathcal{O}(H \cap \omega_{c_0}^t)\}) \rightarrow C^{a,j}(\{Y^t, \mathcal{O}(\omega_{c_1}^t)\})$$

such that $(Ef)^t = f^t$ on $Y^t \times (\omega_{c_1}^t \cap H)$. The similar conclusion holds when $C^{a,j}$ in hypotheses and the conclusion is replaced by $C_*^{b,j}$.

Proof. We may assume that H is defined by $z_n = 0$. Fix t . Since $\omega_{c_0}^t$ is relatively compact in D^t , then $\omega_{c_1}^t$ is relatively compact in $\omega_{c_2}^t$ for $c_1 < c_2 \leq c_0$. Also, ω_c^t is a pseudoconvex domain for each $c < c_0$, if it is non-empty.

Fix $c_1 < c_2 < c_3 < c_4 < c_5 < c_0$. Fix t_0 . When t is close to t_0 , we have

$$\omega_{c_1}^t \subset\subset \omega_* \subset\subset \omega_{c_2}^{t_0} \subset\subset \omega_{c_3}^{t_0} \subset\subset \omega_{c_4}^{t_0} \subset\subset \omega_{c_5}^{t_0} \subset\subset \omega_{c_0}^t.$$

Here ω_* is strictly pseudoconvex domain with C^2 boundary and it depends only on t_0 .

Note that $(\overline{\omega_{c_4}^{t_0}} \cap H) \times \overline{\Delta}_\epsilon \subset \omega_{c_5}^{t_0}$ for some $\epsilon > 0$. Let $\chi(z)$ be a smooth function that is equal to 1 on $(\overline{\omega_{c_3}^{t_0}} \cap H) \times \overline{\Delta}_{\epsilon/2}$ and has compact support in $(\omega_{c_4}^{t_0} \cap H) \times \Delta_\epsilon$. Consider

$$g^t(y, z) = \begin{cases} f^t(y, z') \bar{\partial}_z(\frac{1}{z_n} \chi(z)), & z \in (\omega_{c_4}^{t_0} \cap H) \times \Delta_\epsilon \setminus H \times \{0\}, \\ 0, & z \in (\omega_{c_3}^{t_0} \cap H) \times \Delta_{\epsilon/2} \cup \{\mathbf{C}^n \setminus (\omega_{c_4}^{t_0} \cap H) \times \Delta_\epsilon\}. \end{cases}$$

When ϵ is sufficiently small, the union of two sets in the above two formulae contains $\overline{\omega_{c_2}^{t_0}}$. Thus we see that $\{g^t\} \in C^{a,j}(\{Y^t \times \omega_{c_2}^t\})$.

We now use the linear $\bar{\partial}$ -solution operator T_{ω_*} and define $u^t(y, z) = T_{\omega_*} g^t(y, z)$. Then $\tilde{f}^t(y, z) := \chi(z) f^t(y, z') - z_n u^t(y, z)$ are holomorphic extensions respectively on $\omega_{c_1}^t$ for t close to t_0 . Clearly, for t near t_0 , we obtain that $\{u^t\}$ and hence $\{\tilde{f}^t\}$ are in $C^{a,j}(\{Y^t \times \omega_{c_1}^t\})$. Furthermore, $\{\tilde{f}^t\} \in C^{a,j}(\{Y^t, \mathcal{O}(\omega_{c_1}^t)\})$. Using a partition of unity $\{\psi_\nu\}$ for $[0, 1]$, we get a desired holomorphic extension

$$\tilde{f}^t(y, z) = \sum \psi_\nu(t) (\chi_\nu(z) f^t(y, z') - z_n u_\nu^t(y, z)).$$

That is that, on $\omega_{c_1}^t \cap H$, we have $\tilde{f}^t(y, z) = f^t(y, z')$. □

Let H be a complex subspace of \mathbf{C}^n . If $\{\omega^t\}$ is a continuous family of domains ω^t in \mathbf{C}^n , let $C_H^{b,i}(\{Y^t, \mathcal{O}(\omega_{c_0}^t)\})$ denote the space of $\{f^t\} \in C^{b,i}(\{Y^t, \mathcal{O}(\omega_{c_0}^t)\})$ such that $f^t|_H = 0$.

Lemma 5.5. *Let $D^t, \varphi^t, \omega^t, \omega_c^t$ be as in Lemma 5.4. Let H be defined by $z_1 = \dots = z_\ell = 0$. Let $0 < c_1 < c_0$. Assume that $H \cap \omega_{c_1}^t \neq \emptyset$, for all $t \in [0, 1]$. Suppose that $a' \leq a$, $i \leq j$, and $i \leq a$. There exist linear maps*

$$T_m: C_H^{a',i}(\{Y^t, \mathcal{O}(\omega_{c_0}^t)\}) \rightarrow C^{a',i}(\{Y^t, \mathcal{O}(\omega_{c_1}^t)\})$$

such that $f^t(y, z) = \sum_{m=1}^\ell T_m^t f(y, z) z_m$. The similar conclusion holds when $C^{a,j}$ is replaced $C_*^{b,j}$ for any $b \geq 0$.

Proof. Apply induction on ℓ . For $\ell = 1$, take $f_1^t(y, z) = f^t(y, z)/z_1$. Away from $z_1 = 0$, f_1 clearly has the desired smoothness. To check the smoothness near $z_1 = 0$, we note that $\overline{\Delta}_\delta \times (\overline{\omega_{c_1}^t} \cap H) \subset \omega_{c_0}^{t_0}$ if δ and $|t - t_0|$ are sufficiently small. For z near $(0, p_2, \dots, p_n) \in \overline{\omega_{c_0}^t} \cap H$, we have

$$f_1^t(y, z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1|=\delta, |\zeta_2-p_2|=\delta, \dots, |\zeta_n-p_n|=\delta} \frac{f^t(y, \zeta)}{\zeta_1 \prod (\zeta_i - z_i)} d\zeta_1 \cdots d\zeta_n.$$

It is clear that $\{f_1^t\} \in C^{a,i}(\{Y^t, \mathcal{O}(\omega_{c_1}^t)\})$.

Assume that the lemma holds for $\ell-1$. Then $f^t(y, 0, z') = \sum_{j=2}^{\ell} g_j^t(y, z')z_j$. By Lemma 5.4, we extend $\{g^t\}$ to $\{\tilde{g}^t\} \in C^{a,i}(\{Y^t, \mathcal{O}(\omega_{c_2}^t)\})$ such that $\{\tilde{g}^t\}$ is still of class $C^{a,j}$. Here $c_1 < c_2 < c_0$. Define $\tilde{f}^t(y, z) = f^t(y, z) - \sum_{j=2}^{\ell} \tilde{g}_j^t(y, z)z_j$. Then $\tilde{f}^t(y, z) = 0$ when $z_1 = 0$. So $\tilde{f}^t(y, z) = z_1 \tilde{g}_1^t(y, z)$ and $\{\tilde{g}_1^t\} \in C^{a,i}$. \square

Theorem 5.6 (Hefer's theorem with multi-parameters). *Let r^t, D^t, U^t be as in Theorem 5.1. Let $0 < c_1 < c_0$. There is a continuous linear map*

$$W: C^{a+1,j}(\{Y^t, \mathcal{O}(D_{c_0}^t)\}) \rightarrow C^{a+1,j}(\{Y^t, \mathcal{O}(D_{c_1}^t \times D_{c_1}^t)\})$$

so that $w = Wf$ satisfies $f^t(y, \zeta) - f^t(y, z) = \sum_{i=1}^n w_i^t(y, \zeta, z)(\zeta_i - z_i)$.

Proof. On $Y^t \times D^t \times D^t$ we consider $F^t(y, \zeta, z) = f^t(y, \zeta) - f^t(y, z)$. Now $\omega_c^t \times \omega_c^t$ is define by $\phi^t(z, w) := \max\{\varphi^t(z), \varphi^t(w)\} < c$, and ϕ^t is plurisubharmonic. Apply linear change of coordinates

$$(5.15) \quad L: \tilde{\zeta} = \zeta - z, \quad \tilde{z} = z.$$

Set $\tilde{D}^t = L(D^t \times D^t)$, $\tilde{f}^t(y, \tilde{\zeta}, \tilde{z}) = f^t(y, \tilde{\zeta} + \tilde{z}, \tilde{z})$, and $\tilde{\varphi}^t(\tilde{\zeta}, \tilde{z}) = \phi^t(\tilde{\zeta} + \tilde{z}, \tilde{z})$. Now L , defined by (5.15) identify $C^{a+1,j}(\{Y^t, \mathcal{O}(D^t \times D^t)\})$ with $C^{a+1,j}(\{Y^t, \mathcal{O}(\tilde{D}^t \times \tilde{D}^t)\})$. Replacing D^t, φ^t, f^t by $\tilde{D}^t, \tilde{\varphi}^t, \tilde{f}^t$ respectively in the last two lemmas, we obtain the conclusion. \square

Theorem 5.7 (Decomposition of Henkin-Ramírez functions). *Under the hypotheses of Theorem 5.1, we have additionally $\Phi^t(\zeta, z) = (\zeta - z) \cdot w^t(\zeta, z)$ with*

$$\{w_i^t\} \in C^{a+1,j}(\{D_{\delta_1}^t \setminus D_{-\epsilon}^t, \mathcal{O}(D_{\epsilon/4}^t)\}).$$

Analogously, if $\{r^t\} \in C_*^{b+2,j}(\{U^t\})$, then $\{w_i^t\} \in C_*^{b+1,j}(\{D_{\delta_1}^t \setminus D_{-\epsilon}^t, \mathcal{O}(D_{\epsilon/4}^t)\})$.

Proof. Consider the family of holomorphic functions

$$\{\Phi^t\} \in C^{a+1,j}(\{D_{\delta_1}^t \setminus D_{-\epsilon}^t, \mathcal{O}(D_{\epsilon/2}^t)\}).$$

By Theorem 5.6, we have $\Phi^t(\zeta, z) - \Phi^t(\zeta, \tilde{z}) = \tilde{w}^t(\zeta, \tilde{z}, z) \cdot (\tilde{z} - z)$ for $\zeta \in D_{\delta_1}^t \setminus D_{-\epsilon}^t$ and $\tilde{z}, z \in D_{\epsilon/2}^t$. Then $\tilde{z} = \zeta$ and $w^t(\zeta, z) = \tilde{w}^t(\zeta, \zeta, z)$ are the desired functions. \square

The above theorem does not require that $\nabla r \neq 0$ on ∂D , that is, ∂D might not be smooth. When in additionally ∂D^t is smooth, we can use the Leray map $w(\zeta, z)$ to formulate homotopy formulae. We take the Leray maps

$$g^1(\zeta, z) = w^t(\zeta, z), \quad g^0(\zeta, z) = \bar{\zeta} - \bar{z}.$$

We construct Ω^0, Ω^{01} via g^0 , and g^0, g^1 respectively. The following theorem is a direct consequence of the classical homotopy formula and the solution operator of Lieb-Range.

Theorem 5.8. *Let $a, b \in \overline{\mathbf{R}}_+$ and let $j \in \overline{\mathbf{N}}$ with $j \leq a$. Let $\{U^t\}$ be a continuous family of domains with total space \mathcal{U} , and let $\{\omega^t\}$ be a continuous family of domains with $\omega^t \subset U^t$. Let $\{r^t\}$ be of class $C^{a+2,j}(\mathcal{U})$ (reps. $C_*^{b+2,j}(\mathcal{U})$). Suppose that r^t is strictly plurisubharmonic on ω^t . Let D^t be relatively compact open sets in U^t . Suppose that $\partial D^t =$*

$\{z \in \omega^t: r^t(z) = 0\}$, $r^t > 0$ on $U^t \setminus \overline{D^t}$, and $\partial r^t \neq 0$ on ∂D^t . Let f^t be a $(0, q)$ form on D^t of class $C^{1,0}(\mathcal{D}) \cap C^{0,0}(\overline{\mathcal{D}})$. Then

$$f^t = \overline{\partial} T_{q-1}^t f + T_q^t \overline{\partial} f, \quad 1 \leq q \leq n,$$

$$T_{q-1}^t f = - \int_{\partial D^t} \Omega_{0,q-1}^{01} \wedge f^t + \int_{D^t} \Omega_{0,q-1}^0 \wedge f^t.$$

Here $g^1 = \Phi^t = w^t \cdot (\zeta - z)$ is given by Theorem 5.7 and $g^0 = \overline{\zeta} - \overline{z}$. Assume further that $\overline{\partial} f^t = 0$ on D^t . Then $\overline{\partial} S_q^t f = f$ for

$$S_q^t f = L_q^t E f + K_q^t \overline{\partial} E f, \quad 1 \leq q \leq n,$$

$$L_q^t E f = \int_{U^t} \Omega_{0,q-1}^0 \wedge E^t f, \quad K_q^t \overline{\partial} E f = \int_{U^t \setminus D^t} \Omega_{0,q-1}^{01} \wedge \overline{\partial}_\zeta E^t f.$$

Here $\{E^t f\}$ is a Seeley extension such that for each t , $E^t f$ has compact support in $\mathcal{U}(\overline{D^t})$.

The proof for strictly convex case of Proposition 4.6 can be modified easily to obtain the following.

Theorem 5.9. *Let $1 \leq q \leq n$. Let $j, k \in \overline{\mathbf{N}}$ with $j+1 \leq k$. Let $\{D^t\}$, with total space \mathcal{D} , be a smooth family of strictly pseudoconvex domains D^t with $C^{k+1,j}$ boundary. There exist $\overline{\partial}$ -solution operators S_q for $(0, q)$ -forms on D^t such that for $k' \leq k, j' \leq j, k' < \infty, j' < \infty$,*

$$|S_q f|_{\mathcal{D}; k'+1/2, j'} \leq C_{k'}(|r|_{k'+2})|f|_{\mathcal{D}; k', j'}.$$

Here the constant $C_k(|r|_{k+2})$ depends on norm of $|r|_{k+2}$.

6. THE $\overline{\partial}$ -EQUATION FOR VARIABLE DOMAINS OF HOLOMORPHY

In this section, we present a parametrized version of Oka-Weil approximation in Proposition 6.5. We then obtain interior regularity for a continuous family of domains of holomorphy that admits a continuous family of plurisubharmonic uniform exhaustion functions in Theorem 6.7. As applications we then solve a parametrized version of Levi-problem in Theorem 6.9 and a parametrized version of Cousin problems in Theorem 6.10. The formulation and the solutions of the Cousin problems lead us to study functions defined on general open sets of $\mathbf{C}^n \times [0, 1]$ that are total spaces of $\{D^t\}$ with D^t being possibly empty.

It is a standard fact that for a domain Ω in \mathbf{C}^n which is not the whole space, D is pseudoconvex if and only if $-\log \text{dist}(z, \partial\Omega)$ is a plurisubharmonic function on Ω . Therefore, we have the following.

Lemma 6.1. *Let $\Gamma^t: \overline{D} \rightarrow \overline{D^t}$ be a homeomorphism for each $t \in [0, 1]$, where D, D^t are bounded domains in \mathbf{C}^n . If $\Gamma \in C^{0,0}(\overline{D})$ and D^t are domains of holomorphy, then $\{-\log \text{dist}(z, \partial D^t)\} \in C^{0,0}(\mathcal{D})$ is a family of plurisubharmonic uniform exhaustion functions on D^t .*

Let $\{D^t\}$ be a continuous family of domains in \mathbf{C}^n , i.e. $\mathcal{D} = \cup D^t \times \{t\}$ is open in $\mathbf{C}^n \times [0, 1]$. If K is a subset of \mathcal{D} , we define

$$K^t = \{z: (z, t) \in K\}.$$

By Lemma 4.9, K is compact, if and only if K^t are compact and $\{K^t\}$ is upper semi-continuous.

Let $P^0(D)$ denote the set of continuous plurisubharmonic functions on $D \subset \mathbf{C}^n$, and let $P(D)$ denote the set of plurisubharmonic functions on D . For a subset K of $D \subset \mathbf{C}^n$, the $P(D)$ hull of K , denoted by \widehat{K}_D^P , is the set of $z \in D$ satisfying

$$\varphi(z) \leq \sup\{\varphi(w) : w \in E\}, \quad \forall \varphi \in P^0(D).$$

Proposition 6.2. *Let $\{D^t\}$ be a continuous family of domains in \mathbf{C}^n with total space \mathcal{D} . Let $\{K^t\}$ be a family of subsets K^t of D^t with a compact total space \mathcal{K} . Let φ^t be plurisubharmonic functions on D^t with $\{\varphi^t\} \in C^{0,0}(\mathcal{D})$. Let $\epsilon > 0$. There exists a family $\{\tilde{\varphi}^t\} \in C^{\infty,\infty}(\mathcal{D})$ of functions such that $\tilde{\varphi}^t$ are plurisubharmonic on ω^t , the total space ω of $\{\omega^t\}$ is an open neighborhood of \mathcal{K} , and*

$$0 \leq \tilde{\varphi}^t(z) - \varphi^t(z) < \epsilon, \quad \forall z \in K^t.$$

Proof. To apply smoothing for the variable domains, we first extend $\{D^t\}$ to a larger family of domains as follows. We set $D^t = D^0, K^t = K^0$ for $t < 0$ and $D^t = D^1, K^t = K^1$ for $t > 1$. We define by

$$\varphi_0^t(z) = \varphi^{-t}(z), \quad t < 0; \quad \varphi_0^t(z) = \varphi_0^1(z), \quad t > 1.$$

Then φ_0^t is plurisubharmonic on D^t . Let $\chi(t)$ be a non-negative smooth function with compact support such that $\chi(t) = 1$ near 0 and $\int \chi(t) dt = 1$. Let $\tilde{\chi}(z) = c_n \chi(|z|)$ such that $\int \tilde{\chi}(z) dV(z) = 1$. Set $\chi_\delta(t) = \delta^{-1} \chi(\delta^{-1}t)$ and $\tilde{\chi}_\delta(z) = \delta^{-2n} \tilde{\chi}(\delta^{-1}z)$. Consider

$$\varphi_\epsilon^t(z) = \iint \varphi_0^s(\zeta) \chi_\epsilon(t-s) \tilde{\chi}_\epsilon(z-\zeta) ds dV(\zeta).$$

Note that \mathcal{K} is compact. The sub-mean value property holds for φ_ϵ^t near K^t for all t when ϵ is sufficiently small. Therefore, each φ_ϵ^t is plurisubharmonic near K^t for $0 \leq t \leq 1$. Using a partition of unity and further smoothing, we can achieve $\{\varphi^t\} \in C^{\infty,\infty}(\mathcal{D})$. \square

Proposition 6.3. *Let $\{D^t\}$ with total space \mathcal{D} be as in Proposition 6.2. Suppose that there exist plurisubharmonic uniform exhaustion functions φ_0^t on D^t with $\{\varphi_0^t\} \in C^{0,0}(\mathcal{D})$. Let $\{K^t\}$ be a family of sets with total space \mathcal{K} being a compact subset of \mathcal{D} and let ω be an open neighborhood of \mathcal{K} in \mathcal{D} . Assume that each K^t is $P(D^t)$ convex. There exist strictly plurisubharmonic uniform exhaustion functions φ^t on D^t satisfying the following:*

- (i) $\{\varphi^t\} \in C^{\infty,\infty}(\mathcal{D})$.
- (ii) $\varphi^t < 0$ on K^t and $\varphi^t > 1$ on $D^t \setminus \omega^t$.

Proof. We first prove it when K^t and ω^t are empty for all t . Since $\{(z, t) : \varphi_0^t(z) \leq 0\}$ is compact, adding a constant we may assume that $\varphi_0^t \geq 2$ on D^t . Consider sub-level sets

$$E_m^t = \{z \in D^t : \varphi_0^t(z) \leq m\}.$$

By assumption, the total space \mathcal{E}_m of the family is compact. By Proposition 6.2, we find $\{\varphi_1^t\} \in C^{\infty,\infty}(\mathcal{D})$ such that for each t , $0 \leq \varphi_1^t - \varphi_0^t < 1/4$ on E_2^t and φ_1^t is plurisubharmonic on E_2^t . Set $\hat{\varphi}_1^t = \varphi_1^t$. Let χ_2 be a C^∞ convex function such that $\chi_2(s) = 0$ for $s < 5/4$ and $\chi_2''(s) > 0$ for $s > 5/4$. Take $\{\phi_2^t\} \in C^{\infty,\infty}$ such that ϕ_2^t is plurisubharmonic on E_3^t and $0 \leq \phi_2^t - \varphi_0^t < 1/4$ on E_3^t . Let $\hat{\varphi}_2^t = \hat{\varphi}_1^t + C_2 \chi_2 \circ \phi_2^t$. When C_2 is sufficiently large, $\hat{\varphi}_2^t$ is strictly plurisubharmonic on E_3^t and $\hat{\varphi}_2^t > 1$ on $E_3^t \setminus E_2^t$. Inductively, we find plurisubharmonic ϕ_m^t

such that $|\phi_m^t - \varphi_0^t| < 1/4$ on E_{m+1}^t with $\{\phi_m^t\} \in C^{\infty,\infty}$. Take a smooth convex function χ_m such that $\chi_m(s) = 0$ for $s < m + 1/4$ and $\chi_m''(s) > 0$ for $s > m + 1/4$. Choose C_m sufficiently large such that $\hat{\varphi}_m^t = \hat{\varphi}_{m-1}^t + C_m \chi_m \circ \phi_m^t$ is plurisubharmonic on E_{m+1}^t and $\hat{\varphi}_m^t > m$ on $E_{m+1}^t \setminus E_m^t$. Then $\hat{\varphi}^t = \lim_{m \rightarrow \infty} \hat{\varphi}_m^t$ satisfies all the conditions.

We now deal with the general case. Fix t_0 . We first consider the case where K^{t_0} is empty. By the upper semi-continuity of $\{K^t\}$, we know that K^t is empty when t is close to t_0 . So the above argument shows there are plurisubharmonic functions φ^t defined for t in a neighborhood of t_0 such that $\varphi^t > 1$ on $D^t \setminus \omega^t$ while the family $\{\varphi^t\}$ has class $C^{\infty,\infty}$. We now consider the case where K^{t_0} is non-empty. Since K^{t_0} is $P(D^{t_0})$ convex, we can find a continuous plurisubharmonic function ϕ^{t_0} on D^{t_0} such that

$$(6.1) \quad \max_{K^{t_0}} \phi^{t_0} < 0 < 1 < \min_{E_2^{t_0} \setminus \omega^{t_0}} \phi^{t_0}.$$

Let $L^{t_0} = \max_{E_2^{t_0}} \phi^{t_0}$. Then $L^{t_0} > 0$. Define

$$\varphi^t(w) = \begin{cases} \max(L^{t_0} \varphi_0^t(w), \phi^{t_0}(w)) & \text{if } w \in E_2^t, \\ L^{t_0} \varphi_0^t(w) & \text{if } w \in D^t \setminus E_2^t. \end{cases}$$

Suppose that t is close to t_0 . We want to show that φ^t is well-defined and plurisubharmonic on D^t , and $\{\varphi^t\} \in C^{0,0}(\mathcal{D})$. The E_2^t is contained in D^{t_0} since $\{E_2^t\}$ is upper semi-continuous. Hence φ^t is well-defined. At $w \in \partial E_2^t$, we have

$$L^{t_0} \varphi_0^t(w) = 2L^{t_0} > \phi^{t_0}(w).$$

This shows that $\{\varphi^t\}$ is in $C^{0,0}$ for t close to t_0 . That φ^t is plurisubharmonic on D^t follows from the definition and the inequality we just proved. We can also see that $\{\varphi^t\} \in C^{0,0}$. By the upper semi-continuity of $\{K^t\}$ and $\{E_2^t \setminus \omega^t\}$, we conclude that

$$(6.2) \quad \varphi^t < 0 \quad \text{on } K^t, \quad \varphi^t > 1 \quad \text{on } E_2^t \setminus \omega^t$$

for t close to t_0 . (Of course, the first inequality is vacuous if K^t is empty.) By Proposition 6.2, we may assume that $\{\varphi^t\} \in C^{\infty,\infty}(\{D^t\})$. Again, $\{\varphi^t\} \in C^{0,0}$ is a family of plurisubharmonic uniform exhaustion functions on D^t .

We now find a finite open covering $\{I_\alpha; \alpha \in A\}$ of $[0, 1]$, and continuous plurisubharmonic functions φ_α^t such that the above holds for $\varphi^t = \varphi_\alpha^t$ and $t \in I_\alpha$. Let $\{\chi_\alpha\}$ be a partition of unity subordinate to $\{I_\alpha\}$ with $\chi_\alpha \geq 0$. Then $\varphi^t = \sum \chi_\alpha(t) \varphi_\alpha^t$ is plurisubharmonic on D^t and (6.2) holds for all t . We can repeat the above smoothing and approximation arguments for φ^t . The proof is complete. \square

We need the following parametrized version of the Oka-Weil approximation theorem, which is crucial in solving the $\bar{\partial}$ -equation for variable domains of holomorphy. Let us first state the following elementary result which follows from the Morse lemma.

Lemma 6.4. *Let r be a real function on a domain D in \mathbf{R}^N that has no degenerate critical point. Assume that K is a compact subset of D such that r does not attain any local maximum value at each point in K . For any $\delta > 0$ there exists $\epsilon > 0$ such that for each $z \in K$ there exists ζ such that*

$$r(\zeta) - r(z) > \epsilon, \quad |\zeta - z| < \delta.$$

In other words, the lemma concludes that the value of r must increase by a fixed amount from a point in K within a fixed distance.

Proposition 6.5. *Let $\{D^t\}$ and \mathcal{D} be as in Proposition 6.2. Suppose that there are plurisubharmonic uniform exhaustion functions φ^t on D^t with $\{\varphi^t\} \in C^{0,0}(\mathcal{D})$. Fix $\ell \leq j$ and $\ell < \infty$. Let $\{\omega^t\}$ a continuous family of domains ω^t such that each ω^t contains K_0^t for $K_c^t := \{z \in D^t : \varphi^t(z) \leq c\}$. Let $\{f^t\} \in C_*^{0,j}(\{\omega^t\})$ be a family of holomorphic functions f^t on ω^t . Let $\epsilon > 0$. There are holomorphic functions g^t on D^t such that $\{g^t\} \in C_*^{0,j}(\mathcal{D})$ and*

$$|g - f|_{\mathcal{K}_0; 0, \ell} := \max_{i \leq \ell, t \in [0, 1]} \max_{z \in K_0^t} |\partial_t^i \{(g^t - f^t)(z)\}| < \epsilon.$$

Proof. We follow the method of approximation by using Leray formula. We say that assertion $A_c(\{f^t\})$ holds if for each $\epsilon > 0$ and each finite $\ell \leq j$ there exists a family of functions g^t satisfying the following:

- (i) There is a family $\{g^t\} \in C_*^{0,j}(\{K_c^t\})$ of holomorphic functions g^t defined near K_c^t , i.e. $g^t \in \mathcal{O}(K_c^t)$.
- (ii) $|g - f|_{\mathcal{K}_0; 0, \ell} < \epsilon$.

By assumptions, A_0 holds for all $\{f^t\}$. Let c_* be the supremum of c such that A_c holds for all $\{f^t\}$.

Let us first show that $c_* = \infty$. Assume for the sake of contradiction that c_* is finite. By assumption, we know that $c_* > 0$. We may further assume that

$$(6.3) \quad f^t \in \mathcal{O}(K_{95c_*/100}^t), \quad \{f^t\} \in C_*^{0,j}(\{K_{95c_*/100}^t\}).$$

We find a finite open covering $\{I_i\}_{i=1}^m$ of $[0, 1]$ and $t_i \in I_i$ and $\delta_i > 0$ which have the following properties: For real linear functions L_i with $|L_i| < \frac{1}{200} \min\{1, c_*\}$ on $K_{c_*+9}^{t_i}$, define

$$r_i := \varphi^{t_i} * \chi_{\delta_i} + L_i \in C^\infty(K_{c_*+9}^{t_i}),$$

$$\max\{|r_i(z) - \varphi^{t_i}(z)| : z \in K_{c_*+9}^{t_i}\} < \frac{1}{100} \min\{1, \epsilon_*\}.$$

Here $\chi_\delta(z) = \delta^{-2n} \chi(\delta^{-2n} z)$, $\chi \geq 0$ is a smooth function with compact support in \mathbf{C}^n , and $\int \chi = 1$. For suitable $\delta_i > 0$ and L_i , the r_i are strictly plurisubharmonic and have no degenerate critical point on $K_{c_*+9}^{t_i}$. Furthermore, if

$$\Omega_c^i := \{z \in D^{t_i} : r_i(z) < c\}, \quad E_c^i := \{z \in D^{t_i} : r_i = c\}, \quad L_c^i = \Omega_c^i \cup E_c^i,$$

and $t \in I_i$, then

$$(6.4) \quad K_c^t \subset \Omega_{c+c_*/100}^i, \quad L_c^i \subset K_{c+c_*/100}^t, \quad \forall c \leq c_* + 8.$$

We choose $d_0 > 0$ such that

$$|\partial_\zeta^2 r_i - \partial_z^2 r_i| < \frac{\lambda_0}{C_n}, \quad \text{for } |\zeta - z| < d_0.$$

We know that there exists $\lambda_0 > 0$ such that

$$\sum_{j,k} \frac{\partial^2 r_i}{\partial \zeta_j \partial \bar{\zeta}_k} s_j \bar{s}_k \geq \lambda_0 |s|^2.$$

Define

$$d = \min \{d_0, \text{dist}(L_{c+1}^i, E_{c_*+9}^i) : 8c_*/10 \leq c \leq c_* + 6, 1 \leq i \leq m\}.$$

Then $d > 0$. We now apply Theorems 5.1 and 5.7 for a fixed domain as follows. For all t we take $\omega^t = \Omega_{c_*+9}^i$, $r^t = r_i - c$, and $\delta_0 = 1$. By (5.5), we find $\epsilon_0 \in (0, 1)$ such that for any c satisfying $8c_*/10 \leq c \leq c_* + 6$, there are functions $\Phi_i(\zeta, z)$ and mappings $w_i(\zeta, z)$ that satisfy the properties:

- (i) Let $z \in \Omega_{c+\epsilon_0/4}^i$ and $\zeta \in \Omega_{c+3}^i \setminus \Omega_{c-\epsilon_0}^i$. Then $\Phi_i(\zeta, z)$, $w_i(\zeta, z)$ are holomorphic in z and C^1 in z, ζ , and $\Phi_i(\zeta, z) = w_i(\zeta, z) \cdot (\zeta - z)$.
- (ii) If $\partial\Omega_c^i \in C^1$ and $h \in A(\overline{\Omega_c^i})$, then

$$(6.5) \quad h(z) = c_n \int_{\partial\Omega_c^i} h(\zeta) \frac{w_i(\zeta, z) \cdot d\zeta \wedge (\bar{\partial}_\zeta w_i(\zeta, z) \cdot d\zeta)^{n-1}}{\Phi_i^n(\zeta, z)}, \quad \forall z \in \Omega_c^i.$$

It is crucial from Theorem 5.7 that although Φ_i , w_i depend on c , the ϵ_0 does not depend on c . Also, as mentioned in Remark 4.1, we need the Leray formula (6.5) to carry out the following approximation.

By (5.6) we obtain $|\Phi_i(\zeta, z)| \geq C$ for $r_i(\zeta) \geq c_*/2$ and $r_i(z) \leq c_*/100$. Choose $d_1 \in (0, d)$ such that if $|\tilde{\zeta} - \zeta| \leq d_1$, $r_i(z) \leq c_*/100$, and $r_i(\tilde{\zeta}) \geq r_i(\zeta) \geq c_*/2$, then

$$|\Phi_i(\tilde{\zeta}, z) - \Phi_i(\zeta, z)| \leq \frac{1}{4} |\Phi_i(\tilde{\zeta}, z)|.$$

This shows that $\frac{1}{\Phi_i(\zeta, z)}$, which is holomorphic in z for $r_i(z) < r_i(\zeta)$, can be approximated by polynomials in $\frac{\Phi_i(\zeta, z) - \Phi_i(\tilde{\zeta}, z)}{\Phi_i(\zeta, z)}$ on $r_i(z) \leq c_*/100$ in the super norm. The quotient and $w_i(\zeta, z)$ are holomorphic in z on the domain defined by

$$r_i(z) < r_i(\tilde{\zeta}), \quad r_i(z) < r_i(\zeta) + \epsilon_0/4.$$

Since r is strictly plurisubharmonic in $D_{c_*+9}^i$, it does not attain any local maximum value in the set. By Lemma 6.4, there exists $\epsilon_1 > 0$, depending on d_1 , such that if $r_i(\zeta) \leq c_* + 8$, there exists $\tilde{\zeta}$ such that

$$(6.6) \quad r_i(\tilde{\zeta}) \geq r_i(\zeta) + \epsilon_1, \quad |\tilde{\zeta} - \zeta| \leq d_1.$$

Let $\epsilon_2 = \min(\epsilon_0/4, \epsilon_1, \frac{1}{2})$. Therefore, for each ζ , $\frac{1}{\Phi_i(\zeta, z)}$ can be approximated on the domain defined by $r_i(z) \leq c_*/100$ by holomorphic functions on $r_i(z) < r_i(\zeta) + \epsilon_2$. This is the approximation that we will use in the following argument.

Fix i and assume that $t \in I_i$. We start with a regular value \hat{c}_1 of r_i with $\hat{c}_1 \in (93c_*/100, 94c_*/100)$. By (6.3), we can replace h, c in (6.5) by f^t ($t \in I_i$), \hat{c}_1 respectively. This gives us an integral representation for f^t . We then differentiate f^t and the integrand to get integral representations for $\partial_t^l f^t$ for $l \leq \ell$. For the $\ell + 1$ integrals, we approximate them by Riemann sums to obtain for a given $\epsilon > 0$,

$$(6.7) \quad \sup_{t \in I_i} \sup_{z \in K_0^t} |\partial_t^\ell \{f^t(z) - g_i^t(z)\}| < \epsilon,$$

$$g_i^t(z) = \sum_{m=1}^N \left\{ c_n f^t(\zeta) \frac{P(w_i(\zeta, z), \partial_{\bar{\zeta}} w_i(\zeta, z))}{(\Phi_i)^n(\zeta, z)} \right\} \Big|_{\zeta=\zeta_{i,m}}.$$

Here $P, \zeta_{i,m}$ are independent of $t \in I_i, l = 0, \dots, \ell$, P is a polynomial, and $\zeta_{i,m} \in E_{\hat{c}_1}^i$. By the approximation obtained earlier, we know that each term in the above sum of g_i^t can be approximated on K_0^t by functions that are holomorphic on $L_{\hat{c}_1+\epsilon_2}^i$. This gives us a holomorphic function \tilde{g}_i^t on the above set such that (6.7) holds when g_i^t is replaced by \tilde{g}_i^t . Next, we choose a regular value \hat{c}_2 of r_i with $\hat{c}_1 + \epsilon_2/2 < \hat{c}_2 < \hat{c}_1 + \epsilon_2$. We repeat the argument. We repeat this n_i times, with n_i independent of t via (6.6), to obtain (6.7) for a function g_i^t that is holomorphic on $L_{\hat{c}_1+n_i\epsilon_2/2}^i$. Here n_i is so chosen that $c_*+5 < \hat{c}_1+n_i\epsilon_2/2 < c_*+6$. From (6.4) it follows that g_i^t is holomorphic on $L_{c_*+4}^t$ for $t \in I_i$. Note that $L_{c_*+6} \supset K_{c_*+5}^t$. Using a partition of unity $\{\chi_i(t)\}$, we get the approximation $g^t = \chi_i g_i^t$ so that g^t is holomorphic on $K_{c_*+5}^t$ and $\|f - g\|_{\mathcal{K}_0;0,\ell} < \epsilon$. This shows that $c_* = \infty$.

To finish the poof, let $\epsilon > 0$ and $\{h_i^t\} \in C_*^{0,j}(\{K_i^t\})$ such that $h_i^t \in \mathcal{O}(K_i^t)$ and

$$\|f - h_i\|_{\mathcal{K}_0;0,j} < \frac{\epsilon}{2}, \quad \|h_{i+1} - h_i\|_{\mathcal{K}_i;0,j} < \frac{\epsilon}{2^{i+1}}, \quad i = 0, 1, \dots$$

Then $\lim_{i \rightarrow \infty} h_i^t$ has the desired properties. \square

Remark 6.6. By the Cauchy inequality, if \tilde{h}^t approximates h^t on ω^t , where ω^t are open neighborhoods of compact subsets K^t in $C_*^{0,j}$ norm, then it also approximates in $C^{\infty,j}$.

Theorem 6.7. Let $\{D^t\}$ be a continuous family of domains in \mathbf{C}^n . Let $\{\varphi^t\} \in C^{0,0}(\mathcal{D})$ be a family of plurisubharmonic uniform exhaustion functions φ^t on D^t . Let $0 < \alpha < 1$ and $1 \leq q \leq n$, and let $k, \ell, j \in \overline{\mathbf{N}}$ with $k \geq j$. Let $\{f^t\} \in C_*^{\ell+\alpha,j}(\mathcal{D})$ (resp. $C^{k+\alpha,j}(\mathcal{D})$) be a family of $\bar{\partial}$ -closed $(0, q)$ -forms on D^t . There exist a family of solutions u^t to $\bar{\partial}u^t = f^t$ on D^t so that $\{u^t\}$ is in $C_*^{\ell+1+\alpha,j}(\mathcal{D})$ (resp. $C^{k+1+\alpha,j}(\mathcal{D})$).

Proof. Let $K_m^t = \{z \in D^t: \varphi^t(z) \leq m\}$. Denote by \mathcal{K}_m the total space of $\{K_m^t\}$. We will first find u_m^t such that $\bar{\partial}u_m^t = f^t$ near K_m^t ; more precisely $\bar{\partial}u_m^t = f^t$ on $\tilde{\omega}^t$ while $\{\tilde{\omega}^t\}$ is a continuous family of domains of which the total space contains \mathcal{K}_m . Furthermore, $\{u_m^t\}$ is of class $C_*^{\ell+1+\alpha,j}(\mathcal{D})$. Fix t_0 and assume that t is sufficiently close to t_0 . We know that K_{m+1}^t contains $\overline{K_m^{t_0}}$. Take D^* such that D^* has a C^∞ strictly pseudoconvex boundary. Moreover, K_{m+1}^t contains $\overline{D^*}$ and D^* contains $\overline{K_m^t}$. Let T_{D^*} be a solution operator via the homotopy formula. Let $u_m^t = T_{D^*}f^t$. Then $\{u_m^t\}$ is in $C_*^{\ell+1+\alpha,j}(\{K_m^t\})$ for t close to t_0 . Using a partition of unity $\{\chi_i\}$ on $[0, 1]$, we can find u_m^t such that for each t , $\bar{\partial}u_m^t = f^t$ near $\overline{K_m^t}$. Using cut-off, we may further achieve $\{u_m^t\} \in C_*^{\ell+1+\alpha,j}(\mathcal{D})$.

We now assume that $q > 1$. Then $\bar{\partial}(u_1^t - u_2^t) = 0$ near $\overline{K_1^t}$. By the above arguments, we can find v_1^t such that $\bar{\partial}v_1^t = u_1^t - u_2^t$ near $\overline{K_1^t}$ and $\{\bar{\partial}v_1^t\} \in C_*^{\ell+1+\alpha,j}(\mathcal{D})$. We take $\hat{u}_1^t = u_1^t$ and $\hat{u}_2^t = u_2^t + \bar{\partial}v_1^t$. Then $\hat{u}_2^t = \hat{u}_1^t$ near $\overline{K_1^t}$, $\{\hat{u}_2^t\} \in C_*^{\ell+1+\alpha,j}(\mathcal{D})$, and $\bar{\partial}\hat{u}_2^t = \bar{\partial}u_2^t = f^t$ near $\overline{K_2^t}$. Inductively, we have $\bar{\partial}(\hat{u}_j^t - u_{j+1}^t) = 0$ near $\overline{D_j^t}$. We find v_j^t such that $\bar{\partial}v_j^t = \hat{u}_j^t - u_{j+1}^t$ near $\overline{K_j^t}$, and $\{\bar{\partial}v_j^t\} \in C_*^{\ell+1+\alpha,j}(\mathcal{D})$. Set $\hat{u}_{j+1}^t = u_{j+1}^t + \bar{\partial}v_j^t$. Then $\hat{u}_{j+1}^t = \hat{u}_j^t$ near $\overline{K_j^t}$.

Assume now that $q = 1$. If j is finite, we take $j_\ell = j$; otherwise, we take a sequence of integers j_ℓ tending to ∞ . Analogously, we take a sequence of integers k_ℓ tending to k . Let $\hat{u}_1^t = u_1^t$. Then $u_2^t - u_1^t$ is holomorphic near $\overline{K_1^t}$. By Proposition 6.5, we can find holomorphic functions h_1^t on D^t such that $\{h_1^t\} \in C_*^{\infty,j}$ and

$$|\hat{u}_1 - u_2 - h_1|_{C_*^{\ell_1+1+\alpha,j_1}(\mathcal{K}_1)} < 1/2.$$

Let $\hat{u}_2^t = u_2^t + h^t$. We still have $\bar{\partial}\hat{u}_2 = \bar{\partial}u_2^t = f^t$. Inductively, we find holomorphic functions h_ℓ^t on D^t such that $\{h_\ell^t\} \in C_*^{\infty,j}$ and

$$|\hat{u}_m - u_{m+1} - h_m|_{C_*^{\ell_m+1+\alpha,j}(\mathcal{K}_m)} < \frac{1}{2^m}.$$

Here $\mathcal{K}_m = \{(z, t) : \varphi^t(z) \leq m\}$. We then define $\hat{u}_{m+1} = u_{m+1} + h_m$. Using the Cauchy estimates, we verify that $\{\hat{u}_m^t\}$ converges to $\{\hat{u}^t\}$ in $C_*^{\ell_m+1+\alpha,j}(\mathcal{D})$.

Analogously, we can verify $C^{k+1+\alpha,j}(\mathcal{D})$ regularity of the solutions. \square

Definition 6.8. Let $\{D^t\}$ be a continuous family of domains in \mathbf{C}^n . Let $A_j(\mathcal{D})$ denote the set of families $\{f^t\}$ of holomorphic functions f^t on D^t with $\{f^t\} \in C_*^{0,j}(\mathcal{D})$. Let $\{E^t\}$, with total space \mathcal{E} , be a family of subsets E^t of D^t , the $A^j(\mathcal{D})$ -hull $\{\hat{E}^t\}$ of $\{E^t\}$ is defined by its total space

$$\hat{\mathcal{E}} = \left\{ (z, t) \in \mathcal{D} : |f^s(z)| \leq \sup_{(w,s) \in \mathcal{E}} |f^s(w)|, \forall \{f^t\} \in A^j(\mathcal{D}) \right\}.$$

We say that $\{E^t\}$ is $A^j(\mathcal{D})$ convex if $\hat{E}^t = E^t$ for all t .

As an application of Proposition 6.5, we solve the following version of the Levi problem for domains with parameter.

Theorem 6.9. Let $\{D^t\}$ be a continuous family of domains in \mathbf{C}^n . Let $\{\varphi^t\} \in C^{0,0}(\mathcal{D})$ be a family of plurisubharmonic uniform exhaustion functions φ^t on D^t . Then $\{K_c^t\}$ is $A^j(\mathcal{D})$ convex for all $c \in \mathbf{R}$, where $K_c^t \subset D^t$ is defined by $\varphi^t \leq c$.

Proof. By Proposition 6.3, we may assume that $\{\varphi^t\} \in C^{\infty,\infty}(\mathcal{D})$ and φ^t are strictly plurisubharmonic on D^t . It suffices to show that $(\hat{K}_c^t)^t$ is contained in $K_{c_1}^t$, if $c_1 > c$.

Fix $p \in D^s$ with $\varphi^s(p) = c_1 > c$. Choose a real linear function L such that $\phi^s = \varphi^s + L$ has only discrete critical points in D^s . Let \tilde{D}_c^s be defined by $\phi^s < c$. We also choose L so small on $K_{c_2}^s$ that for some $c' > c$, the $\tilde{D}_{c'}^s$ contains K_c^s .

Let $F(w)$ be the Levi polynomial of φ^s at $w_0 = p^s$. Then

$$\varphi^{t_0}(w) > \varphi^{t_0}(w_0) + 2 \operatorname{Re} F(w) + c|w - w_0|^2.$$

Choose $\delta > 0$ sufficiently small. For $\varphi^{t_0}(w) \leq c' + \delta$ and $|w - w_0| \geq \epsilon$, we have

$$(6.8) \quad 2 \operatorname{Re} F(w) < \delta - c\epsilon^2.$$

Let $\chi(w)$ be a smooth function supported in $B_{2\epsilon}$ such that $\chi = 1$ on B_ϵ . We consider

$$u(w) = \chi(w - w_0)e^{LF(w)} - v(w).$$

We want u to be holomorphic on $K_{c'+\delta}^s$ by solving $\bar{\partial}v_L = g_L(w) := \bar{\partial}(\chi(w - w_0)e^{LF(w)})$. We have $g_L(w) = e^{LF(w)}\bar{\partial}\chi(w - w_0)$, which is zero on B_ϵ . Thus (6.8) implies that

$$|g_L|_0 := \sup_{K_{c'+\delta}^s} |g_L| \leq Ce^{L(\delta-2\epsilon^2)}.$$

Take $0 < \delta < c\epsilon^2/2$ such that $D_* := K_{c'+\delta}^s$ has smooth boundary. Thus $|g_L|_0 \rightarrow 0$ as $L \rightarrow +\infty$. We now solve the $\bar{\partial}$ -equation on D_* by using the homotopy formula on D_* to get the estimate

$$|v_L|_0 \leq C'|g_L|_0,$$

which is uniform in L . Fix L such that $C'Ce^{L(\delta-2\epsilon^2)} < 1/4$. On K_c^s , we get $|u| < 1/4$. Also $|u(w_0)| \geq 1 - |v_L(w_0)| > 3/4$. Note that u is holomorphic on $K_{c'+\delta}^s$. Fix $c < c'' < c' + \delta/2$. Using a cut-off function $\chi(t)$ such that $0 \leq \chi \leq 1$ and $\chi(s) = 1$. We obtain $\{u^t = \chi(t)u\}$ such that each u^t is holomorphic on $K_{c''}^t$. Moreover

$$|u^t| < 1/4 \quad \text{on } K_c^t, \quad |u^s(w_0)| > 3/4.$$

By Proposition 6.5, we can replace u^t by \tilde{u}^t which is holomorphic on D^t such that the above still holds for \tilde{u}^t , while $\{\tilde{u}^t\} \in A^j(\mathcal{D})$. Therefore, (p, s) is not in the $A^j(\mathcal{D})$ hull of K_c . \square

We now consider Cousin problems with parameter. We formulate the problems and its solution as follows. Here it is more convenient to formulate the problems and find the solutions, by identifying a continuous family $\{D^t\}$ with its open total space \mathcal{D} in $\mathbf{C}^n \times [0, 1]$ and by allowing some D^t to be empty. We also identify a family $\{f^t\}$ of functions f^t on D^t with a function $(x, t) \rightarrow f^t(x)$ on \mathcal{D} .

Theorem 6.10. *Let $0 < \alpha < 1$. Let $j, k \in \overline{\mathbf{N}}$ with $k \geq j$ and $k > 0$. Let $\{D^t\}$ be a continuous family of domains in \mathbf{C}^n . Suppose that D^t admit plurisubharmonic uniform exhaustion functions φ^t with $\{\varphi^t\} \in C^{0,0}(\mathcal{D})$. Suppose that $\{\mathcal{D}_a : a \in A\}$ is an open covering of \mathcal{D} , where \mathcal{D}_a is the total space of $\{D_a^t\}$. Let $\{f_{ab}^t\}$ be a family of functions such that each f_{ab}^t is holomorphic on $D_a^t \cap D_b^t$. Assume that $\{f_{ab}^t\} \in C_*^{0,j}(\mathcal{D}_a \cap \mathcal{D}_b)$.*

(i) (First Cousin problem.) Assume that for all $a, b, c \in A$, $f_{ab}^t = -f_{ba}^t$, and

$$f_{ab}^t + f_{bc}^t = f_{ac}^t, \quad \text{on } D_a^t \cap D_b^t \cap D_c^t.$$

There exist families $\{f_a^t\} \in C^{\infty,j}(\mathcal{D}_a)$ of holomorphic functions f_a^t on D_a^t such that $f_a^t - f_b^t = f_{ab}^t$.

(ii) (Second Cousin problem.) Suppose that each f_{ab}^t does not vanish on $D_a^t \cap D_b^t$. Suppose that for all a, b, c , $f_{ab}^t = (f_{ba}^t)^{-1}$, and

$$f_{ab}^t f_{bc}^t = f_{ac}^t, \quad \text{on } D_a^t \cap D_b^t \cap D_c^t.$$

There exists a family $\{f_a^t\}$ of nowhere vanishing holomorphic functions f_a^t on D_a^t such that $\{f_a^t\} \in C^{\infty,j}(\mathcal{D}_a)$ and $f_a^t (f_b^t)^{-1} = f_{ab}^t$, provided there exists a family $\{g_a^t\} \in C^{0,0}(\mathcal{D}_a)$ of functions g_a^t vanishing nowhere on D_a^t such that $g_a^t (g_b^t)^{-1} = f_{ab}^t$ for all a, b .

Proof. Note that when each \mathcal{D}_a is open in \mathcal{D} . When all $f_{a,b}^t$ are holomorphic on $D_a^t \cap D_b^t$, the Cauchy formula implies that $\{f_{ab}^t\} \in C^{\infty,j}(\mathcal{D}_a \cap \mathcal{D}_b)$ for $\{f_{ab}^t\} \in C_*^{0,j}(\mathcal{D}_a \cap \mathcal{D}_b)$.

First, we consider the first Cousin problem. By assumption, $\{\mathcal{D}_a\}$ is an open covering of \mathcal{D} . We choose a partition of unity $\{\varphi_\nu\}$ that is subordinate to the covering $\{\mathcal{D}_a\}$. More precisely, $\varphi_\nu \in C_0^\infty(\mathcal{D}_{e_\nu})$, all φ_ν but finitely many of them, vanish identically on any compact subset of \mathcal{D} , and $\sum \varphi_\nu = 1$ on \mathcal{D} . Set $\varphi_\nu^t(z) = \varphi_\nu(z, t)$ and

$$g_a^t(z) = \sum \varphi_\nu^t(z) f_{e_\nu a}^t(z), \quad \forall z \in D_a^t.$$

Here $\varphi_\nu^t(z) f_{e_\nu a}^t(z) = 0$ if z is not in $D_{e_\nu}^t \cap D_a^t$. We can verify that $\{g_a^t\} \in C^{\infty,j}(\mathcal{D}_a)$ and $g_a^t - g_b^t = f_{ab}^t$. Thus $\bar{\partial} g_a^t = \bar{\partial} g_b^t$ on $D_a^t \cap D_b^t$. This shows that $\tilde{g}^t := \bar{\partial} g_a^t$ is well-defined on D^t . Also $\{\tilde{g}^t\} \in C^{\infty,j}(\mathcal{D})$. By Theorem 6.7 we can find $\{u^t\} \in C^{\infty,j}(\mathcal{D})$ such that

$\bar{\partial}g_a^t = \bar{\partial}u^t$. Now $f_a^t = g_a^t - u^t$ becomes holomorphic on D_a^t and $\{f_a^t\}$ is of class $C^{\infty,j}(\mathcal{D}_a)$, while $f_a^t - f_b^t = f_{ab}^t$.

For the second Cousin problem, we assume that there exists $\{g_a^t\} \in C^{0,0}(\mathcal{D}_a)$ such that $f_{ab}^t = g_a^t(g_b^t)^{-1}$. We first consider the case that each D_a is simply connected. We can write $g_a^t = e^{\log g_a^t}$ with $\{\log g_a^t\} \in C^{0,0}(\mathcal{D}_a)$. Let $h_{ab}^t = \log g_a^t - \log g_b^t$. We want to show that $\{h_{ab}^t\}$ is in $C^{\infty,j}(\mathcal{D}_a \cap \mathcal{D}_b)$. Indeed, $e^{h_{ab}^t} = f_{ab}^t$. Locally, we get $h_{ab}^t(z) = \log f_{ab}^t(z) + 2\pi i m(z, t)$ with $m(z, t) \in \mathbf{Z}$. By continuity, we conclude that m is locally constant. This shows that $\{h_{ab}^t\}$ is in $C^{\infty,j}(\mathcal{D}_a \cap \mathcal{D}_b)$. By the solution of the first Cousin problem, we find $\{h_a^t\} \in C^{\infty,j}(\mathcal{D}_a)$ such that h_a^t is holomorphic on D_a^t , $e^{h_a^t - h_b^t} = f_{ab}^t$, and

$$(6.9) \quad h_a^t - h_b^t = \log g_a^t - \log g_b^t.$$

When \mathcal{D}_a are not simply connected, we apply a refinement $\{\tilde{\mathcal{D}}_\beta: \beta \in B\}$ to the open covering of $\{\mathcal{D}_a: a \in A\}$ such that each $\tilde{\mathcal{D}}_\beta$ is a simply connected open subset of some \mathcal{D}_a , while $\bigcup \tilde{\mathcal{D}}_\beta = \mathcal{D}$. By (6.9), we find $\{\tilde{f}_\beta^t\} \in C^{\infty,j}$ such that, for $\tilde{\mathcal{D}}_\alpha \subset \mathcal{D}_a$,

$$h_\alpha^t - h_\beta^t = \log g_\alpha^t - \log g_\beta^t, \quad \log g_\alpha = \log g_a|_{\tilde{\mathcal{D}}_\alpha}.$$

Then $h_\alpha^t = h_\alpha^t$ is well-defined on D_a^t and $\{h_a^t\}$ provides the desired solutions. \square

Analogously, we verify the following result.

Theorem 6.11. *Let $0 < \alpha < 1$. Let $j, k \in \overline{\mathbf{N}}$ with $k - 1 \geq j$. Let $\{D^t\}$ be a continuous family of uniformly bounded strongly pseudoconvex domains in \mathbf{C}^n of $C^{k+1,j}$ boundary. Suppose that $\{\mathcal{D}_a: a \in A\}$ is an open covering of $\overline{\mathcal{D}}$, where \mathcal{D}_a is the total space of $\{D_a^t\}$. Let $\{f_{ab}^t\} \in C^{k+1/2,j}(\mathcal{D}_a \cap \mathcal{D}_b)$ be a family of functions f_{ab}^t that are holomorphic in the interior of $D_a^t \cap D_b^t$ and satisfy Theorem 6.10 (i) (resp. (ii)). There exists a family $\{f_a^t\} \in C^{k+1/2,j}(\mathcal{D}_a)$ satisfying Theorem 6.10 (i) (resp. (ii)).*

Proof. In the proof of previous theorem, we have $\tilde{g}^t = \bar{\partial}g_a^t$ and $\{\tilde{g}^t\} \in C^{\infty,j}(\mathcal{D})$. When considering boundary, we can only claim $\{\varphi_\nu^t\} \in C^{k,j}(\{D_{e_\nu}^t\})$ obtained by parameterizing $\{\overline{D^t}\}$ via embeddings $\{\Gamma^t\} \in C^{k+1,j}(\overline{\mathcal{D}})$. Thus, we have $\{\tilde{g}^t\} \in C^{k,j}(\overline{\mathcal{D}})$. With $k - 1 \geq j$, using Theorem 4.10, we solve $\bar{\partial}u^t = \tilde{g}^t$ with $\{u^t\} \in C^{k+1/2,j}(\overline{\mathcal{D}})$. \square

When $n = 1$, using Theorem 4.5 we get a more precise result.

Theorem 6.12. *Let $0 < \alpha < 1$. Let $k, j \in \overline{\mathbf{N}}$ with $k \geq j$. Let $\{D^t\}$ be a smooth family of bounded domains in \mathbf{C} with $C^{k+\alpha,j} \cap C^{1+\alpha,j}$ boundary. If $\{f_{ab}^t\}$ in Theorem 6.11 are in $C^{k+\alpha,j}(\mathcal{D}_a \cap \mathcal{D}_b)$ for all a, b , then the solutions $\{f_a^t\}$ are in $C^{k+\alpha,j}(\mathcal{D}_a)$.*

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